Siegel measures after Veech

Abstract

In these notes we summerize the proof of Theorem 0.12 Part 1 of Veech's paper "Siegel measures". It states that in the presence of spectral gap, if the Siegel transform has image in L^2 and if there are absolute bounds for both small and large balls, then one can count almost everywhere.

1 Siegel measures

Definitions and the 2 basic properties Let $\mathcal{M}$ be the set of Borel measures ν on $\mathbb{R}^n$ for which with $N_\nu(R) = \nu(B(R))$ we have

$$M(\nu) = \sup_{0<R<\infty} \frac{N_\nu(R)}{R^n} < \infty.$$

We can equip $\mathcal{M}$ with the $C_c(\mathbb{R}^n)$ weak*-topology. There is a $G = \mathrm{SL}_n(\mathbb{R})$ action, which maps $\mathcal{M}$ to $\mathcal{M}$, and is continuous.

Definition 1.1. A Siegel measure is an G-invariant G-ergodic probability measure on $\mathcal{M}$.

Definition 1.2. The Siegel transform is defined by duality, for $\phi : \mathbb{R}^n \to \mathbb{R}$ set

$$\hat{\phi}(\nu) = \nu(\phi) = \int_{\mathbb{R}^n} \phi(x) d\nu(x)$$

is a function $\hat{\phi} : \mathcal{M} \to \mathbb{R}$.

Theorem 1.3. *For any Siegel measure μ, we have that*

$$\hat{}: \mathcal{L}^1(\mathbb{R}^n, m) \to \mathcal{L}^1(\mathcal{M}, \mu)$$

where m is the Lebesgue measure on $\mathbb{R}^n$.

Theorem 1.4. *For any Siegel measure μ there exists a finite constant c_μ such that for any integrable function ϕ Siegel summation holds*

$$\mu(\hat{\phi}) = c_\mu m(\phi)$$

Proof. Restricting to $\phi \in C_c(\mathbb{R}^n)$, consider $\Psi(\phi) = \mu(\hat{\psi})$, which is a positive bounded functional by Theorem 1.3, and thus defines a measure. By assumption on μ, it must be $\mathrm{SL}_n(\mathbb{R})$-invariant. Noting that $\mathbb{R}^n$ has pricisely two G-orbits, it follows G-measures on $\mathbb{R}^n$ are linear combination of the Dirac measure at 0 and the Lebesgue measure

$$\Psi(\phi) = a\phi(0) + bm(\phi).$$

Use the assumption on $\nu \in \mathcal{M}$ (namely $M(\nu) < \infty$, here in the limit as $R \to 0$) implies that for any sequence $\phi_k \in C_c(\mathbb{R}^n)$ such that $\phi_k \leq \chi_{B(1/k)}$ and $\phi(0) = 1$ we have $\hat{\phi}(\nu) = \nu(\phi_k) \ll \frac{1}{k^n} \to 0$. By dominated convergence, we conclude $\mu(\hat{\phi}) \to 0$ and thus $a = 0$. □

Corollary 1.5.

$$\int_{\mathcal{M}} \frac{N_\nu(R)}{R^n} d\mu(\nu) = c(\mu)\omega_n$$

where ω_n is the Lebesgue volume of the n-dimensional unit ball.

Proof. Take $f = \chi_{B(R)}$ then $\hat{f} = N_\nu(R)$ and apply Siegel's formula. □

Remark 1.6. Analogously to Siegel's formula, one can also show that

$$\int_{\mathcal{M}} \left| \frac{N_\nu(R)}{R^n} - c(\mu)\omega_n \right| d\mu(\nu) = o(R),$$

see Remark 5.21 in Veech.

Theorem 1.7. *If the action of $\mathrm{SL}_n(\mathbb{R})$ has a spectral gap on $L^2(\mathcal{M}, \mu)$ and if the Siegel transform satisfies that for all $\phi \in C_c(\mathbb{R}^n)$ one has in fact $\hat{\phi} \in L^2(\mathcal{M}, \mu)$ (instead of just $L^1(\mathcal{M}, \mu)$) then*

$$\left| \frac{N_\nu(R)}{R^n} - c(\mu)\omega_n \right| = o(R) \text{ for } \mu \text{ -a.e. } \nu.$$

Outline for the proof for Theorem 1.3 Let us restrict to $n = 2$.

$$G = \mathrm{SL}_2(\mathbb{R}) \quad K = \mathrm{SO}_2(\mathbb{R}) \quad U = \left\{ u_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \right\} \quad N = U^T = \{n_t\} \quad A^+ = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \text{ for } \lambda > 0 \right\}$$

Step 0. Reduction to χ_B To check integrability, it is sufficient to check this for $\chi_{B(R)}$ where by linearity and positivity as for any $\phi \in C_c(\mathbb{R}^n)$ we have $\phi \leq c\chi_{B(R)}$ for some $c, R > 0$. In fact, it is sufficient to show for χ_B with $B = B(1)$: Assuming we know that $\hat{\chi_B} \in \mathcal{L}^1(\mu)$ for all μ, we can define a homothety $T_\lambda \phi(x) = \lambda^n \phi(\lambda x)$. Then $T_\lambda \phi$ is supported on B if ϕ is supported on $B(\lambda)$. Thus $T_\lambda \phi$ is integrable for any Siegel measure. Noting that if μ is Siegel, then also $T_{\lambda^{-1}*}\mu$ is a Siegel measure (G commutes with homothety),

$$\mu(\phi) = T_{\lambda^{-1}*}\mu(T_\lambda \phi) < \infty.$$

Step 1. An ergodic Theorem For G acting ergodically on (X, μ) for any $f \in \mathcal{L}^p(X, \mu)$, $1 \leq p < \infty$

$$K \star \delta_a \star f \to \mu(f) \text{ in } L^p(X, \mu) \text{ as } a \to \infty$$

using the notation

$$(K \star \delta_a \star f)(x) = \int_K (k\bullet a\bullet f)(x) dk = \int_K f(akx) dk.$$

where dk is the normalzied probability Haar measure on K. We will reduced this to a second, more classical ergodic theorem

$$T^{-1} \int_0^T n_t \bullet f dt \to \mu(f) \text{ in } L^p(X, \mu).$$

$p = 2$ is actually sufficient for our purposes.

Step 1.5. Integrability Criterion We deduce that if for $f \geq 0$

$$\limsup_a K \star \delta_a \star f \leq \infty \quad \mu-\text{a.e.}$$

then $f \in \mathcal{L}^1(X, \mu)$. Indeed, if $f_T = \min(f, T) \in \mathcal{L}^\infty$ then we can choose a subsequence $a_n \to \infty$ such that μ-a.e.

$$\mu(f_T) = \lim K \star \delta_{a_n} \star f_T \leq \limsup K \star \delta_a \star f < \infty$$

and by monotone convergence also $\mu(f) < \infty$.

Step 2. Bounding $K \star \delta_a \star \hat{\phi}$ Note that by Fubini,

$$K \star \delta_a \star \hat{\phi} = \int_{\mathbb{R}^2} \int_K \phi(akx) dk d\nu(x).$$

The goal is to show that for $a = (\lambda, \lambda^{-1})$,

$$\int_{\mathbb{R}^2} \int_K \chi_B(akx) dk d\nu(x) = \frac{2}{\pi} \int_0^1 \frac{N_\nu(\lambda\tau)}{(\lambda\tau)^2} (1 - \tau^2)^{-\frac{1}{2}} d\tau + \mathcal{O}(M(\nu)\lambda^{-\frac{4}{3}})$$

which implies $\limsup K \star \delta_a \star \hat{\phi} \leq M(\nu)$ and thus Theorem 1.3.

2 Proof of the Ergodic Theorem (Step 1)

By the Howe-Moore theorem, we know that N acts ergodically on $L^2(X, \mu)$. By amenability, we have an ergodic theorem

$$T^{-1} \int_0^T n_t \bullet f dt \to \mu(f) \text{ in } L^2(X, \mu).$$

Denote the average measure by ξ_T. There is a map

$$k : N \to K \quad n \mapsto \exp(\log n - \log n^T)$$

recalling that the Lie algebra of U is the span of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and that of K is spanned by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Moreover, k is defined everywhere as log is polynomial on U. For $a = \mathrm{diag}(\lambda, \lambda^{-1})$, set $k(n, \lambda) = k(ana^{-1})$. Note that $ana^{-1} \to e$ as $a \to \infty$, and thus $k(n, \lambda) \to e$ as $\lambda \to \infty$. Since

$$a^{-1} k(n, \lambda) a = \exp(\log n - a^{-2} \log n^T a^2) = ng$$

where $g = g(n, \lambda)$ satisfies $\|g - e\|_\infty = \mathcal{O}(\|n\|_\infty \lambda^{-2})$ using the BCH-formula. Since the action in the strong operator norm is continuous, we note that

$$g \cdot f = f + \mathcal{O}(\delta(f, n, \lambda)) \text{ in } L^2(X, \mu)$$

and using that the action is unitary,

$$\begin{aligned} K \star \delta_a \star f &= \int_K ka \cdot f dk = \int_K \int_N kk(n, \lambda) a \cdot f \; d\xi_T dk \\ &= \int_K \int_N kang \cdot f \; d\xi_T dk = \mathcal{O}(\delta(f, n_T, \lambda)) + \int_K \int_N kan \cdot f \; d\xi_T dk \end{aligned}$$

From the ergodic theorem for N, $\int_N n \cdot f \; d\xi_T \to \mu(f)$ in L^2 as $T \to \infty$, and using (once more) that the action is unitary, also $\int_K \int_N kan \cdot f \; d\xi_T dk \to \mu(f)$ in L^2 also $T \to \infty$ independently of a. Finally let $\lambda \to \infty$, then $T \to \infty$.

3 Proof of Upper bound regarding spherical integrals (Step 2)

Put

$$\mathcal{C}\psi(a) = \int_K \psi(ake_2) dk = \frac{1}{2\pi} \int_0^{2\pi} \psi(\lambda_1 \sin\theta, \lambda_2 \cos\theta) d\theta$$

where $a = \operatorname{diag}(\lambda_1, \lambda_2)$ with $\lambda_1 > \lambda_2 > 0$ is not necessarily of determinant one. For $\psi = \chi_B$ (or any other even function)

$$\mathcal{C}\psi(a) = \frac{1}{\pi} \int_{S^{1,+}} \psi(\lambda_1 x, \lambda_2 y) \frac{dx}{y} = \frac{1}{\lambda_1 \pi} \int_{L_{\lambda_1} S^{1,+}} \psi(x, \lambda_2 y) \frac{dx}{y} \leq \frac{1}{\lambda_1} \operatorname{diam}(\psi) \|\psi\|_\infty = \frac{2}{\lambda_1}$$

proven by drawing the picture.

Rewriting again, introduce

$$F(Ra) = \mathcal{C}\chi_B(Ra)$$

then noting that $N_\nu(R) = \nu(B(R))$ as a commulative distribution is monotone and right semi-continuous so that the Lebesgue-Stieltes integral $dN_\nu(R)$ is well defined,

$$\int_{\mathbb{R}^2} \int_K \chi_B(akx) dk d\nu(x) = \int_{\mathbb{R}^2} \int_K \chi_B(\|x\| ake_2) dk d\nu(x) = \int_{\mathbb{R}^2} F(a\|x\|) d\nu(x) = \int_0^\infty F(Ra) dN_\nu(R).$$

We want to bound this using the integration by parts formula, $d(fg) = fdg + gdf$. Let $a = \operatorname{diag}\left(\lambda, \lambda^{-1}\right) \in A$. We split the integral, into $[0, 2\lambda^{-1}] \cup (2\lambda^{-1}, \infty)$. By above bound on $\mathcal{C}\psi(Ra)$

$$\begin{aligned} \int_0^{2\lambda^{-1}} F(Ra) dN_\nu(R) &\leq \int_0^{2\lambda^{-1}} \frac{2}{R\lambda} dN_\nu(R) = \frac{2}{R\lambda} N_\nu(R) \Big|_0^{2\lambda^{-1}} - \int_0^{2\lambda^{-1}} \frac{-2}{R^2 \lambda} N_\nu(R) dR \\ &\leq M(\nu) 4\lambda^{-2} + M(\nu) 4\lambda^{-2} = 8M(\nu)\lambda^{-2} \end{aligned}$$

We note that $F(Ra) = 0$ if $R\lambda^{-1} > 1$, in particular the integral over $(2\lambda^{-1}, \infty)$ can be restricted to $(2\lambda^{-1}, \lambda)$. Assume $2 < \lambda^2$ in order to have an non-empty interval. We integrate by parts,

$$\int_{2\lambda^{-1}}^{\lambda} F(Ra) dN_\nu(R) = F(Ra) N_\nu(R) \Big|_{2\lambda^{-1}}^{\lambda} - \int_{2\lambda^{-1}}^{\lambda} N_\nu(R) \frac{d}{dR} F(Ra) dR.$$

For the first term we have,

$$F(Ra) N_\nu(R) \Big|_{2\lambda^{-1}}^{\lambda} \leq M(\nu) \left(\lambda^2 F(\lambda a) - 4\lambda^{-2} F(\lambda^{-1} a)\right) = \mathcal{O}(M(\nu)\lambda^{-2})$$

using that $F(\lambda a) = 0$ and that $F(\lambda^{-1} a) \leq 2$.

We have to understand the derivative

$$\frac{d}{dR} F(Ra) = \nabla F(Ra) \cdot a$$

which we claim to be of order

$$-\frac{1}{\lambda R^2} \frac{2}{\pi} (1 - R^2 \lambda^{-2})^{-\frac{1}{2}} \left(1 + \mathcal{O}(\frac{1}{R^2 \lambda^2})\right).$$

Let $b = \mathrm{diag}\,(\lambda_1, \lambda_2)$ then

$$F(b) = \frac{2}{\pi} \arcsin \Lambda(b)$$

where $\Lambda(b) = \left(\frac{1-\lambda_2^2}{\lambda_1^2-\lambda_2^2}\right)^{1/2}$. Indeed, by symmetry we may consider the integral

$$F(b) = \frac{1}{2\pi} \int_0^{2\pi} \chi_B(\lambda_1 \sin\theta, \lambda_2 \cos\theta) d\theta$$

over one quadrant only (say $\theta \in [0, \pi/2]$, changing the weight to $\frac{2}{\pi}$. The integrand then does not vanish for all θ such that $(\lambda_1 \sin\theta)^2 + (\lambda_2 \cos\theta)^2 \leq 1$ or $\sin^2\theta \leq \Lambda(b)^2$ and thus we restrict the integral to $(0, \arcsin \Lambda(b))$. As $\frac{d}{dt} \arcsin t = (1-t^2)^{-1/2}$, we verify that

$$\partial_{\lambda_1} \arcsin \Lambda(b) = \left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 - 1}\right)^{1/2} \partial_{\lambda_1} \left(\frac{\lambda_1^2 - 1}{\lambda_1^2 - \lambda_2^2}\right)^{1/2} = \left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 - 1}\right)^{1/2} (-1)(1-\lambda_2^2)^{1/2} \frac{\lambda_1}{(\lambda_1^2 - \lambda_2^2)^{3/2}}$$

and

$$\partial_{\lambda_2} \arcsin \Lambda(b) = \left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 - 1}\right)^{1/2} \partial_{\lambda_2} \left(\frac{\lambda_1^2 - 1}{\lambda_1^2 - \lambda_2^2}\right)^{1/2} = \left(\frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 - 1}\right)^{1/2} \frac{1-\lambda_1^2}{(1-\lambda_2^2)^{1/2}} \frac{\lambda_2}{(\lambda_1^2 - \lambda_2^2)^{3/2}}$$

so that

$$\nabla F(b) \cdot b = \frac{-1}{(\lambda_1^2 - 1)^{1/2}(1-\lambda_2^2)^{1/2}}.$$

A quick calculation gives that $(\lambda^2 - 1)^{-1/2} = \lambda^{-1} + o(\lambda^{-2})$ so that for $b = (R\lambda, R\lambda^{-1})$ we have

$$\frac{d}{dR} F(Ra) = \frac{1}{R} \nabla F(Ra) \cdot Ra = -\frac{2}{\pi} \frac{1}{R^2 \lambda} (1 + o(R^{-2}\lambda^{-2})) \frac{1}{(1-R^2\lambda^{-2})^{1/2}}.$$

Thus, pluging in, reordering, and substituting $\tau = R\lambda^{-1}$

$$\begin{aligned}
\int_{2\lambda^{-1}}^{\lambda} N_\nu(R) \frac{d}{dR} F(Ra) dR &= \frac{2}{\pi} \int_{2\lambda^{-1}}^{\lambda} N_\nu(R) \frac{1}{(1-R^2\lambda^{-2})^{1/2}} \frac{1}{R^2\lambda} (1 + o(R^{-2}\lambda^{-2})) dR \\
&= \frac{2}{\pi} \int_{2\lambda^{-1}}^{\lambda} \frac{N_\nu(R)}{R^2} \frac{1}{(1-R^2\lambda^{-2})^{1/2}} (1 + o(R^{-2}\lambda^{-2})) \frac{dR}{\lambda} \\
&= \frac{2}{\pi} \int_{2\lambda^{-2}}^{1} \frac{N_\nu(\tau\lambda^{-1})}{(\tau\lambda^{-1})^2} \frac{1}{(1-\tau^2)^{1/2}} (1 + o(\tau^{-2}\lambda^{-4})) d\tau.
\end{aligned}$$

The main term satisfies

$$\frac{2}{\pi} \int_{2\lambda^{-2}}^{1} \frac{N_\nu(\tau\lambda^{-1})}{(\tau\lambda^{-1})^2} \frac{1}{(1-\tau^2)^{1/2}} d\tau = \frac{2}{\pi} \int_0^1 \frac{N_\nu(\tau\lambda^{-1})}{(\tau\lambda^{-1})^2} \frac{1}{(1-\tau^2)^{1/2}} d\tau + \mathcal{O}(M(\nu)\lambda^{-2}).$$

As for the o term, we split the integral into two parts: τ large so that $o(\tau^{-2}\lambda^{-4})$ small and τ small such that the domain of integration is small. Thus, let $1 > \alpha > 0$, to be optimized, and consider the two intervals $(2\lambda^{-2}, (2\lambda^{-2})^\alpha)$ and $((2\lambda^{-2})^\alpha, 1)$. For the latter, we have

$$o(\tau^{-2}\lambda^{-4}) = o\left((\lambda^{-2})^{-2\alpha}\lambda^{-4}\right) = o\left(\lambda^{4(\alpha-1)}\right)$$

and thus the integral over $((2\lambda^{-2})^\alpha, 1)$ is $o\left(M(\nu)\lambda^{4(\alpha-1)}\right)$ On $(2\lambda^{-2}, (2\lambda^{-2})^\alpha)$ we only use that $o(\tau^{-2}\lambda^{-4})$ is bounded, note that the integral is

$$\mathcal{O}\left(M(\nu) \arcsin \tau \Big|_{2\lambda^{-2}}^{(2\lambda^{-2})^\alpha}\right) = \mathcal{O}(M(\nu)\lambda^{-2\alpha})$$

since $\arcsin \tau \approx \tau$ for small τ. Now choose $\alpha = 2/3$ to have $\lambda^{4(\alpha-1)}$ and $\lambda^{-2\alpha}$ of same exponent $-\frac{4}{3}$. This concludes the bound claimed in Step 2.

4 Proof of the Theorem 1.7

We need to collect a couple of tools. We restrict again to $n = 2$.

Effective Decay of Matrix coefficients To say that the action of G on $L^2(\mathcal{M},\mu)$ has a spectral gap implies that there exists a constant $\eta > 0$ such that for any K-invariant $f, f' \in L^2_0(\mathcal{M},\mu)$ of norm one,

$$|\langle g \cdot f, f' \rangle| \leq \lambda^{-\eta}$$

where $g = k_1 a k_2$ with $a = \operatorname{diag}\left(\lambda, \lambda^{-1}\right) \in A^+$. From this, we want to deduce that for f as above

$$\|K \star \delta_a \star f\|_2 = \mathcal{O}(\lambda^{-\xi})$$

where $\xi = \frac{2\eta}{\eta+1}$.

To that end, change the order of integration and G-invariance and K-invariance of the respective Haar measures

$$\|K \star \delta_a \star f\|_2^2 = \int_K dk \int_K dk' \langle ka \cdot f, k'a \cdot f \rangle = \int_K dk \langle a^{-1}ka \cdot f, f \rangle.$$

To apply the bound of matrix coefficients, we have therefore to control $a_k = \operatorname{diag}\left(\lambda_k, \lambda_k^{-1}\right)$, diagonal element in the cartan composition of $a^{-1}ka$. Just as in the previous chapter, we split the integral over K for which $k \approx e$ on which there is no useful bound because also $a^{-1}ka \approx e$ (which only happen for $\|k\| \ll \lambda^{-*}$) and respectively for which $k \in K$ is bounded away from k. We use the Hilbert-Schmidt-norm on $G \subset \operatorname{Mat}_2(\mathbb{R})$

$$\|g\|_{\mathrm{HS}}^2 = \sum \langle e_i, g e_j \rangle^2$$

where e_i is some arbitrary orthonormal basis (with respect to the standard inner product on $\mathbb{R}^2$). This definition is independent of the choice of e_i. It will also important to note that $\|\cdot\|_{\mathrm{HS}}$ is bi-K-invariant, which follows by noting that $\|g\|_{\mathrm{HS}}^2 = \operatorname{tr}(g^T g)$. In particular,

$$\|a^{-1}ka\|_{\mathrm{HS}} \leq 2\lambda_k.$$

If $k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ then

$$\|a^{-1}ka\|_{\mathrm{HS}}^2 = k_{11}^2 + k_{22}^2 + k_{12}^2\lambda^{-4} + k_{21}^2\lambda^4 = 2k_{11} + k_{12}^2\lambda^{-4} + (1-k_{11}^2)\lambda^4 \geq \max(2k_{11}^2, (1-k_{11}^2)\lambda^4)$$

and since $(a^{-1}ka)^{-1} = a^{-1}k^{-1}a$, $(k^{-1})_{11} = k_{11}$ we also get

$$2\lambda_k^{-1} \geq \max(2k_{11}^2, (1-k_{11}^2)\lambda^4)$$

Note that $\{k : (1-k_{11}^2) \leq t\}$ consists of the double cone (in coordinates $k = k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ the usual rotation matrix), those θ for which $(\sin\theta)^2 \leq t$. As for small t, $\arcsin t \approx t$, we have get

$$m_K(\{k : (1-k_{11}^2) \leq t\}) = \frac{1}{2\pi} m_{\mathbb{R}}((\sin\theta)^2 \leq t) = \mathcal{O}(\sqrt{t}).$$

Splitting the integral over K, we have

$$\int_K dk \langle a^{-1}ka \cdot f, f \rangle \leq \mathcal{O}(\sqrt{t}) + \mathcal{O}((\frac{1}{\sqrt{t}}\frac{1}{\lambda^4})^\eta)$$

Take $t = \lambda^\alpha$ and choose α such that both terms are of both exponent, i.e. $\alpha = -8\frac{\eta}{1+\eta}$ and

$$\|K \star \delta_a \star f\|_2^2 = \mathcal{O}(\lambda^{-4\frac{\eta}{1+\eta}})$$

Borel-Cantelli With $f = \hat{\chi_B}(\nu) - c(\mu)\omega_n$, and $\{a_n\}$ such that $\lambda_n^{-4\frac{\eta}{1+\eta}}$ is summable, we have by Borel-Cantelli that $K \star \delta_{a_n} \star \hat{\chi_B}(\nu) \to (\mu)\omega_n$ μ-a.e./ and therefore also the average

$$\frac{2}{\pi}\int_0^1 \frac{N_\nu(\lambda_n\tau)}{(\lambda_n\tau)^2}(1-\tau^2)^{-\frac{1}{2}} d\tau$$

by Step 2. Using the Wiener Tauberian theorem one can show that this implies that also the average

$$\frac{1}{R}\int_0^R \frac{N_\nu(R)}{R^2} dR$$

converges. Finally, one can deduce that also pointwise convergence holds.

To do: Add Wiener-Tauberian Theorem. The last step (Lemma 5.17 in Veech) might also follow from part b) in Rudin's book of this Theorem about slowly oscillating functions.

5 Epilogue: Siegel's application

Let $\Gamma = \mathrm{SL}_n(\mathbb{Z}) < G$ and note that Γe_1 agrees with the set of primitive integer points. Let π the G-invariant map

$$\pi : \Gamma\backslash G \to \mathcal{M} \quad g\Gamma \mapsto \nu_g = \sum_{v\in g\Gamma e_1} \delta_v$$

onto the space of counting measures $\mathcal{M}$ over primitive lattice points. Let μ be the push forward of the Haar measure on $\Gamma\backslash G$, necesssarily ergodic because Haar measure is. Note that the sum excludes the zero vector. Moreover, $N_{\nu_g}(R) \sim \zeta(n)^{-1}\omega_n R^n$, which implies that $M(\nu_g) < \infty$ and thus that μ is a Siegel measure. By the corollary, the asymptotics also show that $c(\mu) = \zeta(n)^{-1}$.

Minkowski - Hlawka Theorem Let B be a star domain. Suppose that for each $\Lambda \in G$, $\Lambda^{-1}B \cap \mathbb{Z}^n \neq \{0\}$ then it must also intersect to primitive points by assumption that B is a star domain. Let $f = \chi_B$ then

$$\sum_{v\in\mathbb{Z}^n \text{ primitive}} f(\Lambda v) = \sum_{v\in\Lambda^{-1}B \text{ primitive}} 1 \geq 1$$

and thus

$$\int_{\mathbb{R}^n} f(x)dx \geq \zeta(n)$$

using Siegel's formula for $\mu = \pi_* m_{\Gamma\backslash G}$ and $c(\mu) = \zeta(n)^{-1}$ as above. Thus we have $\mathrm{vol}(B) \geq \zeta(n)$. This implies, that if for some B we have $\mathrm{vol}(B) < \zeta(n)$, then there must be a set of positive measure on $\mathcal{M}$ for which all elements ν satisfy $\nu(\chi_B) < 1$, thus $\nu(\chi_B) = 0$, that is, there exists lattices which do not intersect B.

If $B = B_r$ is a ball of volume $\sigma_n r^n$, then this says that there is a set of positive measure of lattices with shortest vector greater than $\sqrt[n]{\xi(n)/\sigma_n}$.

For sphere packing, this means the following. Apply the previous to a ball S of volume less then $\zeta(n)$ and take a lattice Λ without intersection. Then $\Lambda + S/2$ are disjoint. Attaching to each fundamental domain of Λ the ball $S/2$, we get an asymptotic density of $\zeta(n)/2^n$.