Graduate Colloquium Vortrag

1 Unitary Representation

Definition 1.1. A unitary representation $(\pi, \mathcal{H})$ of a topological group G is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ and continuous in the sense that for all $v \in \mathcal{H}$ we have $\pi(g_i)v \to \pi(g)v$ if $g_i \to g$.

Examples: Shift and Multiplication

- $\mathrm{SO}(n+1)$ acts on S^n by rotations, $x \in S^n$ (as vector in $\mathbb{R}^{n+1}$) and $\mathrm{SO}(n+1)$, $k \cdot x = kx$ vector matrix multiplication. There is also an representation on the space of functions $f : S^n \to \mathbb{C}$ by $k \cdot f$ defined by $(k \cdot f)(x) = f(k^{-1} \cdot x)$ (to make it still a left-action). To get a unitary representation, we consider $L^2(S^n, \mu)$ with μ an $\mathrm{SO}(n+1)$-invariant measure (e.g. the Lebesgue measure), that is, $k_* \mu = \mu$. Then the *substituion* rule makes this into a unitary representation.
- Of course $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, and this holds more generally for G a topological group and H a closed subgroup such that G/H has an invariant measure μ. Then G acts unitarily on $L^2(G/H, \mu)$.
- Multiplication operator in $L^2(\mathbb{T}, \mu)$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ could also be identified with $[-\frac{1}{2}, \frac{1}{2})$ by χ_m where $\chi_m(x) = e^{2\pi i m x}$, i.e. $m \cdot f = \chi_m f$ is unitary because $|\chi_m|^2 = 1$.
- More generally $\mathbb{Z}^n$ on $\mathbb{T}^n$. In fact, these the spectral theorem for unitary operators says that all $\mathbb{Z}^n$-actions are of that form. More precisely, $(\pi, \mathcal{H})$ a unitary representation of $\mathbb{Z}^n$, $v \in \mathcal{H}$ then there is a finite measure μ_v on $\mathbb{T}^n$ (called **spectral measure**) such that

$$\langle m \cdot v, v \rangle_{\mathcal{H}} = \int \chi_m d\mu_v$$

and for each Borel set B there is a projection $P(B) \in \mathcal{B}(\mathcal{H})$ such that

$$\langle P(B)v, v \rangle_{\mathcal{H}} = \int \chi_B d\mu_v = \mu_v(B).$$

Consider the special case where $\mathbb{H}$ is finite-dimensional and $n = 2$. Then in *short*: $\pi(m)$ are simultaneously diagonalizable say with eigenbasis $\{v_i\}$, and eigenfunctions are homomorphisms $\mathbb{Z}^2 \to S^1$, thus are of the form $\chi_m(s,t) = e^{2\pi i \langle m, (s,t) \rangle}$. Eigenvalue of v_i with respect to $\pi(1,1)$ determines all the others, say $\chi_{(1,1)}(s_i, t_i)$ and thus $\pi(m)$ has eigenvalue $\chi_m(s_i, t_i)$. Spectral measure of v_i is then $\delta_{(s_i, t_i)}$.

Long: Write $m = (x, y) \in \mathbb{Z}^2$ and $\pi(x, y)$. Since $\pi(x, 0)$ is unitary it is diagonalizable (using a unitary matrix, that is, an isometric isomorphism) for each x. But $\pi(x, 0)$ commute for $x \in \mathbb{Z}$ so there are simultaneously diagonalizable. In fact $\pi(x, 0) = \pi(1, 0)^x$ so $\pi(x, 0) \simeq \mathrm{diag}\,(\lambda_1^x, \dots, \lambda_{\dim \mathcal{H}}^x)$ if λ_i are the eigenvalues of $\pi(1, 0)$. I can do the same thing for $\pi(0, y) \simeq \mathrm{diag}\,(\nu_1^x, \dots, \nu_{\dim \mathcal{H}}^y)$ (using the same isomorphism from before by commutativity) and thus $\pi(x, y) = \pi(x, 0)\pi(0, y) = \mathrm{diag}\,(\dots, \chi_m(s, t), \dots)$ if $\chi_m = \chi_m \exp 2\pi i(x s_i + y t_i)$. Then if $v = \sum a_l v_l$ where v_l is an eigenvector (simultaneous for $\pi(1, 0)$ and $\pi(0, 1)$) then $d\mu(s, t) = \sum_i \delta_{s_i}(s)\delta_{t_i}(t)$ is the spectral measure and if $v = v_i$ is an eigenvector (say) $\mu_{v_i} = \delta_{s_i}\delta_{t_i}$ and more generally if $v = \sum a_i v_i$, $\mu_v = \sum |a_i|^2 \delta_{s_i}\delta_{t_i}$ since

$$\langle \pi(x, y)v, v \rangle = \langle \sum a_i \chi_m(s_i, t_i) v_i, \sum a_i v_i \rangle = \sum |a_i|^2 \chi_m(s_i, t_i) = \sum |a_i|^2 \delta_{s_i}(\chi_x)\delta_{t_i}(\chi_y)$$

$P(B)$ is just the obvious projection to $\mathrm{span}\, v_i : (s_i, t_i) \in B$.

Spectral Gap

- $(\pi, \mathcal{H})$ is said to have an **almost invariant vectors** if for any (Q, ϵ) where Q compact and ϵ positive there exists a unit vector v such that $\|g \cdot v - v\| < \epsilon$ for all $g \in Q$.
- $(\pi, \mathcal{H})$ without (honest) invariant vectors is said to have a **spectral gap** if it does not contain almost invariant vectors.
- G is said to have **Property (T)** if all unitary representations without fixed points have a spectral gap. Equivalently: There exists (Q, ϵ) such that if there are (Q, ϵ) almost invariant vectors then there is an invariant vector.

Examples to Spectral Gap

- Using the spectral measure for $\mathbb{Z}^n$-actions one can see that there is is no invariant vector if 0 is not an atom of μ_v a spectral gap if and only if 0 is not in the support of 0. (Else one could just take smaller and smaller characteristic functions over sets on which finitely many characters do no vary much.) Taking the Lebesgue measure, we deduce that $\mathbb{Z}$ has not (T). MUST USE ACTION ON $\mathbb{R}^n$!
- Compact groups have (T) (Take $Q = G$ the Haar measure, average over an almost vector to get an invariant vector)
- $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{Z})$ does not have (T) for many reasons (Free group (but needs lattice thm) and cont homo with dense image also has, Cheeger, Laplace, Unitary dual)

2 Expander

Expanding constant / isoperimetric constant

$$h(G) = \inf_{|A| \leq |V|/2} \frac{|\partial A|}{|A|}$$

also makes sense for G infinite. Here $\partial A = \{v \in V - A : \{v, a\} \in E \text{ for some } a \in A\}$. If G is a Caley Graph (or more general a Schreier graph) with symmetric (generating) set S, $\partial A = \bigcup_{s \in S}(sA - A)$.

Lemma 2.1. *A countable finitely generated group* Γ *is amenable if and only if* $h(\Gamma) = 0$.

Proof. Defining amenability using Foelner's property we have need to compare $|\partial A|$ with the symmetric difference. $|sA \Delta A| = |sA - A| + |s^{-1}A - A|$ and summing $\sum_{s \in S} |sA \Delta A| = 2 \sum_{s \in S} |sA - A|$ which implies

$$|\partial A| = |\bigcup_{s \in S}(sA - A)| \leq \sum_{s \in S} |sA - A| = \tfrac{1}{2} \sum_{s \in S} |sA \Delta A|$$

and

$$\frac{1}{2|S|} \sum_{s \in S} |sA \Delta A| \leq \max_{s \in S} |sA - A| \leq |\bigcup_{s \in S}(sA - A)| = |\partial A|.$$

Foelner states that $\inf_{|A| < \infty} \sup_{s \in Q} \frac{|sA \Delta A|}{|A|} = 0$ for all finite Q but since finitely generated, it suffices for one, taking $S = Q$. Then taking sup is equivalent to taking sums and thus the claim. □

Definition 2.2. $G = (V, E)$ finite graph and enumerate $V = 1, \ldots, n = |E|$. Let G be k-regular (that is, $|\partial\{i\}| = k$ for all $i \in V$). Denote by A the normalized adjacency matrix $A_i j = 1$ if $\{i, j\} \in E$ and else 0. A as an operator on $\ell^2(V)$ has the meaning of averaging:

$$Af(x) = \sum_{x \sim y} f(y).$$

Definition 2.3. One can use a different interpretation fixing some orientation on G, i.e. $e = (e^-, e^+) = (i, j) \in E$ come as ordered tuples. Let $d : \ell^2(V) \to \ell^2(E)$ defined by $df(e) = f(e^+) - f(e^-)$ and let $\Delta : \ell^2(V) \to \ell^2(V)$ defined by $\Delta f = d^* df$. It is uniquely defined by $\langle g, \Delta f \rangle_{\ell^2(V)} = \langle dg, df \rangle_{\ell^2(E)}$.

Proposition 2.4.

$$\Delta = kI - A \text{ and } \Delta f(x) = \sum_{x\sim y}(f(x) - f(y))$$

Remark 2.5. kA is symmetric and has eigenvalues in $[-k, k]$. k corresponds to the constant function on each connected component and is thus simple if G is connected. Thus Δ has eigenvalues in $[0, 2k]$ and positive on $\ell_0^2(V)$ the orthogonal complement to the constant function. The smallest positive eigenvalue λ_1 might be called the **spectral gap**. Or, for the adjency matrix $\{\mu_{n-1} \leq \cdots \leq \mu_1 < k$, $k - \mu_1$ is the spectral gap. Also

$$k - \mu_1 = \lambda_1.$$

Remark 2.6. For an infinite k-regular tree, the spectrum $\sigma(A) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$

Theorem 2.7. *(Alon-Boppana) For any k-regular graph with $|V| = n$, $\lambda_1 \geq \sqrt{k-1} - o_n(1)$.*

Definition 2.8. One can also define Edge expansion

$$h_{\text{edge}} = \inf_{|S|\leq|V|/2} \frac{|E(S, S^c)|}{|S|}$$

where $|E(S, S^c)| = |\{(x, y) \in S \times S^c\}|$ counts the number of edges between S and S^c. Then $h_{\text{edge}} \sim h$ (namely $h \leq h_{\text{edge}} \leq dh$).

Theorem 2.9. *(Cheeger/Buser)(Tanner/Dodziuk/Alon and Milman/Alon)*

$$\frac{\lambda_1}{2} = \frac{k - \mu_1}{2} \leq h_{edge} \leq \sqrt{2k(k - \mu_1)} = \sqrt{2k\lambda_1}.$$

We thus may also write

$$\lambda_1 \ll_d h \ll_d \lambda_1^{1/2}.$$

Remark 2.10. Compare to when Δ is Laplacian on a connected compact Riemannian manifold of dimension n with smallest positive eigenvalue $\lambda_1(M) = \inf\{\|Df\|_2^2/\|f\|_2^2 : f \in C^\infty(M)$ and $\int_M f = 0$. Then Cheeger constant $h(M) = \inf_E \frac{area(E)}{\min(vol(A), vol(B))}$ where the $n-1$-dimensional submanifold E cuts M into A and B satisfies $\lambda_1(M) \geq h(M)^2/4$

Relative Property (T) $H \subset G$ closed and $(\pi, \mathcal{H})$ is said to have a spectral gap with respect to (G, H) if $\mathcal{H}$ has no H-invariant vectors (stronger assumption!) then there are no (G)-almost invariant functions.

Proposition 2.11. Let (S, ϵ) be a Kazhdan pair for $(\mathrm{SL}(2, \mathbb{Z}), \mathbb{Z}^2)$ where $S = \{u^{\pm 1}, n^{\pm 1}, \pm e_1, \pm e_2$ and let $G = (\mathbb{Z}/m\mathbb{Z})^2$ neighbour relation defined by $x \sim y$ if $sx = y$ for some $s \in S$ then $h(G) > \left(\frac{\epsilon}{2}\right)^2$.

Proof. $\mathrm{ASL}(2, \mathbb{Z}^2)$ acting on $\ell^2((\mathbb{Z}/m\mathbb{Z})^2)$ by translation, $(\gamma \cdot f)(x) = f(\gamma^{-1} \cdot x)$ is a unitary representation (because the counting measure on $(\mathbb{Z}/m\mathbb{Z})^2$ is invariant under $\mathrm{ASL}(2, \mathbb{Z})$). We have to consider the orthogonal complement of the space of $\mathbb{Z}^2$-invariant functions - which is just $\ell_0^2((\mathbb{Z}/m\mathbb{Z})^2)$ the integral zero function since $\mathbb{Z}^2$-acts transitively. (S, ϵ) being a Kashdan pair means that for any f with $\|f\|_2 = 1$ there exists $\gamma \in S$ such that $\|\gamma f - f\|_2 \geq \epsilon$. The right choice for showing expansion is to put for $A \sqcup B = (\mathbb{Z}/m\mathbb{Z})^2$ and $|A| = a$, $|B| = b$ the function $f(x) = b$ for $x \in A$ and $f(x) = -a$ for $x \in B$ then $\|f\|_2^2 = ab^2 + a^2b = ab(a + b) = abn$ where $n = m^2$ so $\|\gamma f - f\|^2 > \epsilon^2 abn$. On the other hand

$$\|\gamma f - f\|^2$$

$$= \sum_{x\in A:\gamma^{-1}x\in A} (f(\gamma^{-1}x) - f(x))^2 + \sum_{x\in A:\gamma^{-1}x\in B} (f(\gamma^{-1}x) - f(x))^2$$

$$+ \sum_{x\in B:\gamma^{-1}x\in B} (f(\gamma^{-1}x) - f(x))^2 + \sum_{x\in B:\gamma^{-1}x\in A} (f(\gamma^{-1}x) - f(x))^2$$

$$\sum_{x\in A:\gamma^{-1}x\in A} 0 + \sum_{x\in A:\gamma^{-1}x\in B} (-a - b)^2 + \sum_{x\in B:\gamma^{-1}x\in B} 0 + \sum_{x\in B:\gamma^{-1}x\in A} (b - (-a))^2$$

$$= n^2 \left| \{x \in A : (x, \gamma^{-1}x) \in A \times B\} \bigcup \{x \in B : (x, \gamma^{-1}x) \in B \times A\} \right|$$

$$\leq n^2 2\partial A$$

and therefore

$$\epsilon^2 abn \leq n^2 2\partial A.$$

We are assuming that $a \leq n/2$ we have $b \geq n/2$ and thus

$$\frac{\partial A}{a} \geq \epsilon^2 \frac{bn}{2n^2} \geq \epsilon^2 \frac{1}{4}.$$

Taking the infinum over all A with $a \leq n/2$ we find that

$$h(G) = \inf_{|A| \leq |V|/2} \frac{|\partial A|}{|A|} \geq \left(\frac{\epsilon}{2}\right)^2$$

□

Proposition 2.12. Same assumption as the last Proposition then $\lambda_1 \geq \epsilon^2/2$.

Proof. Based on the fact that

$$\langle \Delta f, f \rangle = \frac{1}{2} \sum_{(x,y)} |f(x) - f(y)|^2$$

and therfore for f an eigenfunction of λ_1 of norm $= 1$ (orthogonal to 1 and thus in ℓ_0^2)

$$\lambda_1 = \frac{1}{2} \sum_{s \in S} \sum_{x \in V} |f(sx) - f(x)|^2 = \frac{1}{2} \sum_{s \in S} \|sf - f\|^2 \geq \max_{s \in S} \|sf - f\|^2 \geq \frac{1}{2} \epsilon^2$$

□

3 Property (T)

Theorem 3.1. *Let k be a local field and and $\mathbb{G}$ a simple (connected) algebraic group defined over k of k-rank ≥ 2. Then $\mathbb{G}(k)$ has Property (T).*

Theorem 3.2. *Let G be a locally compact group and H a closed subgroup such that $H \backslash G$ has a finite invariant regular Borel measure. Then G has (T) if and only if H has (T).*

Theorem 3.3 (Bekka, Mayer)**.** *G a connected Lie group has (T) if and only if there is $\epsilon \geq 0$ such that*

$$\inf\{\langle d\pi(\Delta) f, f \rangle : f \in \mathcal{H}^{\infty,K}\}, \|f\| = 1\}$$

for all unitary representations $(\pi, \mathcal{H})$ of G without fixed vector and K a compact subgroup of K and $\mathcal{H}^{\infty,K}$ the space of K-finite C^∞-vectors.

A spectral gap of $(\pi, \mathcal{H})$ is equivalent with that 0 is not an approximative eigenvalue of $d\pi(\Delta)$.

Theorem 3.4 (of Burger91, Shalom99)**.** *$S = \{u^{\pm 1}, n^{\pm 1}, \pm e_i\}$ are a generating set with Kazdhan constant $1/10$ for $(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z}^2)$.*

Proof. Let $(\pi, \mathcal{H})$ be unitary representation. We proof by contradiction. That is, let π have no invariant vectors but v a unit vector that is ϵ-invariant under S. Restrict π to $\mathbb{Z}^2$ so that for each $v \in \mathcal{H}$ with $\|v\| = 1$ there is a probability measure μ_v on $\mathbb{T}^2$ such that for $n \in \mathbb{Z}^2$ $\langle nv, v \rangle = \mu_v(\chi_n)$ with $\chi_n(x) = \exp(2\pi x \cdot n)$ which defines μ_v uniquely. We note that

- $0 \in \operatorname{Supp} \mu_v$ if and only if there is an $\mathbb{Z}^2$-invariant vector in $\mathcal{H}_v = \overline{\langle nv : n \in \mathbb{Z}^2 \rangle}^{\mathcal{H}}$. We thus assume that $\mu_v(\{0\}) = 0$.
- There is projection valued measure $P : \mathcal{B}(\mathbb{T}^2) \to \mathrm{Proj}(\mathbb{H})$ defined by $\langle P(B)v, v \rangle = \mu_v(B)$
- Uniqueness of μ_v implies that $\mu_v(gB) = \langle g^{-1} P(B) g v, v \rangle$.

We verify the last one: Since G acts unitarily,

$$\langle g^{-1}ngv, v\rangle = \langle ngv, gv\rangle = \mu_{gv}(\chi_n).$$

On the other hand, $g^{-1}ng = g^{-1}n$ where the LHS is a product in $\mathrm{ASL}_2(\mathbb{Z})$ and the RHS is just matrix vector multiplication and defines an element in $\mathbb{Z}^2$. Therefore

$$\langle g^{-1}ngv, v\rangle = \mu_v(\chi_g^{-1}n) = \mu_v(\chi_n(g^{-T}\cdot) = g_*^{-T}\mu_v(\chi_n).$$

and since χ_n defines μ_v uniquely

$$g_*^{-T}\mu_v = \mu_{gv}.$$

We want to show the following two things regarding μ_v which we may identify as a probability measure on $(-\frac{1}{2}, \frac{1}{2}]^2 \setminus \{0\}$.

- Almost invariance under $\pm e_1, \pm e_2$ implies that most mass of μ_v must be contained in $(-\frac{1}{4}, \frac{1}{4}]^2 \setminus \{0\}$
- Almost invariance $u^{\pm 1}, n^{\pm 1}$, and the fact that any $T \in \{u^{\pm 1}, n^{\pm 1}\}$ maps $(-\frac{1}{4}, \frac{1}{4}]^2$ to $(-\frac{1}{2}, \frac{1}{2}]^2$ we find an almost invariant measure on $\mathbb{R}^2 \setminus \{0\}$, of which there is none.

Let $X = (-\frac{1}{4}, \frac{1}{4}]^2$ and $\mathbb{T}^2 = (-\frac{1}{2}, \frac{1}{2}]^2$.

Step 1: $\mu_v(X) \geq 1 - \epsilon^2$ Let $g = e_1$ then combining

$$\|gv - v\|^2 = \int_{\mathbb{T}^2} |e^{2\pi i x} - 1|^2 d\mu_v(x, y) \leq \epsilon^2$$

with

$$|e^{2\pi i x} - 1|^2 \geq 2 \text{ on } \left\{\frac{1}{4} \leq |x| \leq \frac{1}{2}\right\}$$

gives

$$\mu_v(\{x : |x| \geq \frac{1}{4}\}) \leq \epsilon^2/2$$

and similiarly for y.

Step 2: $|\mu_v(gB) - \mu_v(B)| \leq 2\epsilon$ **for** $g = u^{\pm 1}, n^{\pm 1}$

$$|\mu_v(gB) - \mu_v(B)| = |\langle g^{-1}P(B)gv, v\rangle - \langle P(B)v, v\rangle|$$
$$\leq |\langle g^{-1}P(B)gv, v\rangle - \langle g^{-1}P(B)v, v\rangle| + |\langle g^{-1}P(B)v, v\rangle - \langle P(B)v, v\rangle|$$
$$= |\langle g^{-1}P(B)(gv - v), v\rangle| + |\langle P(B)v, (gv - v)\rangle|$$
$$\leq \|g^{-1}P(B)\|\|gv - v\| + \|P(B)\|\|gv - v\| \leq 2\epsilon$$

Step 3: For any finitely additive probability measure on $\mathbb{R}^2 - 0$ **there is** $g = u^{\pm 1}, n^{\pm 1}$ **and a Borel set** B **with** $|\nu(gB) - \nu(B)| \geq 1/4$ Decompose $\mathbb{R}^2 - 0$ into 4 regions formed by rotating the x-axis by 45 degrees and call them A, B, C, D and repeat identifying p with $-p$. Then

$$u(A\bigcup B) \subset A \quad n(A\bigcup B) \subset B \quad u^{-1}(D\bigcup C) \subset D \quad n^{-1}(D\bigcup C) \subset C.$$

But assuming that ν contradicts the assertion we have

$$\nu(B) = \nu(A \sqcup B) - \nu(A) \leq \nu(A \sqcup B) - \nu(u(A\bigcup B)) < 1/4$$

and similiarly for the other sets, so that we conclude the contradiction

$$\nu(A \sqcup B \sqcup C \sqcup D) < 1.$$

Step 1+2+3 Let now $\nu = \mu_v|_X/\mu_v(X)$ the normalized restriction of μ_v to X. First consider $\mu = \mu_v|_X$. By Step 1 we know that $|\mu_v(B) - \mu(B)| \leq \mu_v(X^c) \leq \epsilon^2$. But μ is also almost invariant under $g = u, n$ and there inverses using Step 2:

$$(\mu(gB) - \mu(B)) = (\mu(gB) - \mu_v(gB)) + (\mu_v(gB) - \mu_v(B)) + (\mu_v(B) - \mu(B)) \leq \epsilon^2 + 2\epsilon + \epsilon^2.$$

Therefore, normalizin and using Step 1 once more, $|\nu(gB) - \nu(B)| \leq 2(\epsilon + \epsilon^2)/(1-\epsilon^2) = 0.22/0.99 < 1/4$ which contradicts step 3.

□

4 Hausdorff-Banach-Tarski paradox

Definition 4.1. G group acting on a set X. Subsets A and B of X are said to be (G)-**equidecomposable** if A and B can each be partitioned into the same number of respectively G-congruent pieces, i.e. $A = \sqcup_{i\leq n} A_i$, $B = \sqcup_{i\leq n} B_i$ and there $g_i \in G$ such that $g_i A_i = B_i$. Notation: $A \sim B$.

Definition 4.2. X is G-**paradoxical** if there are disjoint sets A, B of X such that $A \sim X \sim B$.

Remark 4.3. Free Group F Denote by F_x the reduced words beginning by $x \in \{a, b, a^{-1}, b^{-1}\}$. Then $F = F_a \cup aF_{a^{-1}} = F_b \cup bF_{b^{-1}}$ so $F_a \cup F_{a^{-1}} \sim F \sim F_b \cup F_{b^{-1}}$

Free action If F acts on X freely then X is F-paradoxical: Let M be a representatives of F-orbits in X (axiom of choice!). Let $X_x = F_x(M) = \{tm : t \in F_x,\ m \in M$ then X_x are disjoint for $x \in \{a, b, a^{-1}, b^{-1}\}$, i.e. decompose each F-orbit into these 4 suborbits which makes them disjoint since F has no fixed points.

Proposition 4.4. SO(3) contains a free group on two generators.

Proof. Let

$$a = \frac{1}{5}\begin{bmatrix} 3 & -4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } b = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}.$$

It will suffice to calculate modulo 5, and replace a and b with $5a$ respectively $5b$. a, b and their inverse have one-dimensional image under 5. Eigenvectors are $w_a = (2, 1, 0)$ and $w_b = (0, 2, 1)$ and permutation of rows $w_{a^{-1}}$ and $w_{b^{-1}}$ for their inverses. $x \in \{a, b, ...\}$ maps w_y to w_x if $y \neq x^{-1}$. $w_{x^{-1}}$ lies in the kernel of x. The inverses are just row and column rotation and since $(1, 1, 1)$ will be mapped (mod 5) to w_a under a, it is also true that this vector is mapped to w_x under x for all $x \in \{a, b, ...\}$. Let F be the free group with generators α and β and map α to a and β to b. One needs to show that this map is injective, which is equivalent to saying that a reduced word has a unique representation. But if W is a word, $W(1, 1, 1)$ is always non-trivial by the above (indeed, the only happentence of mapping an eigenvector of a different matrix to 0 is when this part was not reduced the first place). □

Proposition 4.5. The sphere S^2 is SO(3)-paradoxical.

Proof. Let F be a free subgroup of SO(3). Since every non-trivial element in SO(3) has exactly two fixed points the set $D = \{x \in S^2 : \gamma x = x : \gamma \in F\}$ is fixed by F-only and $S^2 \setminus D$ is therefore F-paradoxical and SO(3)-paradoxical.

Let ℓ be the line through the origin that misses the countable set D. Let θ be such that for every integer $n \geq 0$, $\rho^n(D) \cap D = \emptyset$ where ρ is the roation around ℓ of angle θ. It exists sicne D is countable. Let $S = \cup_{n=0}^{\infty} \rho^n(D)$ then $S^2 = S \cup S^2 \setminus S \sim \rho(S) \cup S^2 \setminus S = S^2 \setminus D$. The latter was shown to be SO(3)-paradoxical and thus S must be. □

Proposition 4.6. Any two subsets of nonempty interior of S^n are SO(n)-equidecomposable.

Corollary 4.7. *A rotation invariant, finitely additive measure ν on the Lebesgue subsets of S^n, $n > 1$ must be absolutely continuous with respect to the Lebesgue measure λ.*

Proof. Let E be Lebesgue mearuable and $\lambda(E) = 0$. If D_i disc such that $\text{diam}(D_i) \to 0$ then $\nu(D_i) \to 0$ since number of disjoint translation goes to ∞. By previous proposition, $D_i \sim S^n$ and therefore $E \sim T_i$ for some subset $T_i \subset D_i$. That is, there exists g_j with $E = \sqcup E_j$ (measurable b/c Lebesgue measure 0) and $T_i = \sqcup T_{i,j}$ with $E_j = g_j T_{i,j}$. By invariance, and finite additivity of ν therefore $\nu(E) = \sum \nu(E_j) = \sum \nu(T_{i,j}) \to 0$ and thus $\nu(E) = 0$. □

Proposition 4.8. The Lebesgue measure is not the unique invariant mean on S^1. In fact, it is not even the unique absolutely continuous invariant mean of S^1.

Remark 4.9. on(2) is amenable as discrete group. Thus there is an invariant mean on $L^\infty(G)$, which consists of *all* bounded function, and gives a finite additively measure on all subsets of G.

Proposition 4.10. The Lebesgue measure is the unique invariant σ-additive measure on S^n for all n.

Proof. Either uniqueness of Haar measure on $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ or by covering properties of balls (compare argument for interval). □

5 Solutions to the Banach-Ruziewicz Problem

Named after Stefan Banach and Stanisłav Ruziewicz.

Proposition 5.1. $\Gamma < \mathrm{SO}(n+1)$ finitely generated acting on $L^2(S^n, \mu)$ with a spectral gap (and ergodically!) then μ is the unique invariant mean on $L^\infty(S^n)$.

Proposition 5.2. $\Gamma = \mathbb{G}(\mathbb{Z})$ is a lattice in $\prod_{p\in S} \mathbb{G}(\mathbb{Q}_p)$ where $\mathbb{G}$ is a connected semi-simple algebraic $\mathbb{Q}$-group.

5.1 Drinfel'd's proof

Let ν be a mean measure on S^n not proportional to μ the Lebesgue measure. Then any discrete $\Gamma < \mathrm{SO}(n+1)$ acting on $L^2_0(S^n, \mu)$ contains the identity representation weakly. Apply this to

$$D^* \mapsto (D \otimes \mathbb{R})^*/\mathbb{R}^*_+ \simeq \mathrm{SU}(2) \to \mathrm{SO}(n+1).$$

5.2 Margulis' proof

Small G invariant sets on (X, μ) if for any $t > 0$ and $\epsilon > 0$, any $F \subset G$ finite there exists $A \subset X$ measurable such that $\mu(A) \in (0, t]$ and $\mu(gA\Delta A) \leq \epsilon\mu(A)$ for all $g \in F$.

Fact 5.3. Ameanability implies existence of such that, Theorem 2.4 in Counterexamples in ergodic theory and number theory.

Weak containment $\tau \prec (\pi, \mathcal{H})$ iff for all compact $K \subset G$ and every $\epsilon > 0$ there exists $0 \neq x \in \mathcal{H}$ such that $\|\pi(g)x - x\| \leq \epsilon\|x\|$ for every $g \in K$.

Small G-invariant sets implies weak containment of τ in $\lambda|_{L^2_0(X,\mu)}$ Let $\chi_A \in L^2_0(X, \mu)$ projection of the the characteristic function over A. Then

- $\|\chi_A\|^2 = \mu(A)(1 - \mu(A))$
- $\|\chi_A^g - \chi_A\|^2 = \mu(gA\Delta A)$

5.2.1 Propositions

Since ergodicity implies that $\tau \notin \lambda|_{L^2_0(X,\mu)}$ it follows that

Proposition 5.4. If G has (T) and acts ergodically on X then X does not admit almost invariant sets.

Fact 5.5. H countable acting ergodically on (X, μ) then there exists small almost H-invariant sets with respect to μ if and only if μ is NOT the only H-invariant mean on $L^\infty(X, \mu)$.

Proposition 5.6. If G contains a countable subgroup with (T) and G acts ergodically on (X, μ) then μ is the only G-invariant mean on $L^\infty(X, \mu)$.

Thus also shows that for $n > 2$ the Lebesgue measure on torus is only mean invariant under $\mathrm{SL}_n(\mathbb{Z})$. (also true for $n = 2$)

Proposition 5.7. $\mathrm{SO}_n(\mathbb{Z}[\frac{1}{5}])$ is dense and has property (T).

Proof. Both $\mathrm{SO}_n(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{Q}_p)$ have (T) because rank 0 and rank 2 groups have it. Thus the lattice $\mathrm{SO}_n(\mathbb{Z}[\frac{1}{5}])$ has it. It is irreducible and thus by definition dense. □

6 Spectral decomposition of $L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}))$

6.1 Representation theory for PGL_2

P upper triangular matrices, then there are one-dimensional characters χ_s for $s \in \mathbb{C}$ on P defined by

$$\chi_s \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = |ad^{-1}|^s$$

where $|\cdot|$ is either real absolute value or p-adic absolute value. Induction to G, $\mathrm{Ind}_P^G(\chi_s)$ give all **irredicuble unitary representation of class one** (i.e. representations with a K-invariant vector). There are two kinds, the **Principal Series** $\{it : t \in \mathbb{R}\}$ and the **Complementary Series** $\{s : -\frac{1}{2} \leq s \leq \frac{1}{2}\}$ for G_∞ and $\{s + \frac{n\pi i}{\log p} : -\frac{1}{2} \leq s \leq \frac{1}{2},\ n \in \mathbb{Z}\}$ for G_p

If ρ_s is such a representation that $\phi_s(g) = \langle \rho_s(g)v, v\rangle$ for v K-invariant is a bi-K-invariant function called **spherical function**. This one can be considered on G/K which is either the upper half-plane or the $(p+1)$-regular tree. It is an eigenfunction of the Laplacian Δ, $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ with eigenvalues $\lambda(s) = \frac{1}{4} - s^2$ respectilve of $Af(x) = \sum_{x\sim y} f(y)$ with eigenvalues $\lambda(s) = p^{\frac{1}{2}}(p^s + p^{-s})$

Proposition 6.1. $\Gamma < G$ a lattice then ρ_s appears as a subrepresentation of $L^2(\Gamma\backslash G)$ if and only if $\lambda(s)$ is an eigenvalue of Δ resp. A on the surface (resp. graph) $\Gamma\backslash G/K$.

Theorem 6.2. *If Γ' is a congruence subgroup of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ then $\lambda_1(\Gamma'\backslash\mathbb{H}) \geq 3/16$. A congruence subgroup is a lattice of finite index in Γ which contains $\Gamma(m) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}))$ for some $m \neq 0$.* Selberg Conjecture *is the claim that $\lambda_1(\Gamma'\backslash\mathbb{H}) \geq 1/4$ which corresponds to no complementary series.*

The analogue is that $\Gamma\backslash G_p/K$ is a Ramanujan graph ($|\lambda(s)| \leq 2\sqrt{p}$) if and only there is no complementary series in $L^2(\Gamma\backslash G_p)$.

Theorem 6.3 (Deligne's Theorem)**.** *Let $\rho = \otimes\rho_s$ be a irriducible (not one-dimenstional) representation of $\mathrm{PGL}_2(\mathbb{A})$ in $L^2(\mathrm{PGL}_2(\mathbb{Q})\backslash \mathrm{PGL}_2(\mathbb{A}))$ Assume that ρ_∞ is a discerete series representation of $\mathrm{PGL}_2(\mathbb{R})$ then for every finite p, ρ_p is not in the complementary series.*

7 Amenability

7.1 Additive measures and Dual

Σ a σ-algebra. $\mathrm{ba}(\Sigma)$ Banach space of bounded, finitely additive signed measures on Σ and norm being variation, e.g. $|\mu|(E) = \sup_\pi \sum_{A\in\pi} |\mu(A)|$ where π goes over all finite partitions. $\mathrm{ca}(\Sigma) \subset \mathrm{ba}(\Sigma)$ space of countably additive measures. $\mathrm{ba}(\Sigma) = B(\Sigma)^*$ where $B(\Sigma)$ space of bounded measurable function with uniform norm. Now μ fixed Borel (σ-additive) measure, Σ the Borel σ-algebra then $L^\infty(\mu)$ is quotient space of $B(\Sigma)$ with $N_\mu = \{f \in B(\Sigma) : f = 0\mu - a.e.\}$. Then $(L^\infty(\mu))^* = N_\mu^\perp = \{\Lambda \in ba(\Sigma) : \Lambda(f) = 0 \forall f \in N_\mu\}$ which is the space of absolutely continuous finitely additive measures. If the space is σ-finite the $(L^1(\mu))^* = L^\infty(\mu)$ and (by Radon Nikodym) $L^1(\mu)$ the space of absolutely continuous (σ-additive) measures. Inclusion into the bidual $L^1(\mu) \subset (L^\infty(\mu))^*$ is therefore an embedding of ac-measures into ac finitely additive measures.

Mean is a positive linear functional in $(L^1(G))^*$ with underlying measure the Haar measure of G locally compact.

7.2 Amenable: Equivalent Definitions

Mean $\exists$ (left) invariant mean

Irreducible dual $\pi \prec \lambda$ where π irred unitary and λ left regular representation.

Trivial representation $\tau \prec \lambda$ where τ trivial representation

Foelner condition For every compact $F \subset G$ and every $\epsilon > 0$ there exists measurable $U \subset G$ such that $\sup_{g\in F} m(U\Delta gU)/m(U) < \epsilon$.

Kesten condition Left-convolution with a prob meas on G has operator norm 1.