Equidistribution on affine symmetric spaces

1 Sources

- Eskin-Mcmullen - Mixing, Counting, and Equidistribution in Lie Groups
- Schlichtkrull - Hyperfunctions and Harmonic Analysis on Symmetric Spaces
- Knapp - Representation Theory of semisimple groups: Beyond an introduction

2 Affine Symmetric Spaces

Definition 2.1. Let G be a connected semi-simple Lie group with finite center. Let $\sigma : G \to G$ be an involution (i.e. a Lie group automorphism with $\sigma^2 = \mathrm{id}$) and let $H < G$ the fixpoint set of σ. Then G/H is called *affine symmetric space* and H is called a symmetric group.

Recall that G is semisimple if its Lie algebra $\mathfrak{g}$ is a direct sum of simple Lie algebras. The differential of σ at the identity gives a Lie automorphism that is an involution, also denoted by σ. Any linear involution is diagonalizable - splitting into ± 1-eigenspaces. This decomposition keeps holding in the group level, where however, only one eigenspace is a lie algebra. For a decomposition $g = hb$ we can then write $\sigma(g) = hb^{-1}$.

Example 2.2. Let $G = \mathrm{SL}_n(\mathbb{R})$ the group of $n \times n$-matrices of det 1 and $\sigma(g) = g^{-T}$ inverse transpose. $\mathrm{stab}(\sigma) = \mathrm{SO}_n(\mathbb{R})$. More generally, any classical Lie group that is closed under transposition. For an involution σ with H compact, G/H defines a Riemannian symmetric space.

Example 2.3. $G \times G/G$ where G is diagonally embedded comes from the convolution $\sigma(g, h) = \sigma(h, g)$. Eg $\{M \in \mathrm{Mat}_{dd}(\mathbb{R}) | \det M = a\} = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})/\Delta\,\mathrm{SL}_2(\mathbb{R})$

Example 2.4. $\mathrm{SL}_2(\mathbb{R})/A$ where A the diagonal group coming from $\sigma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$.

Example 2.5. We let $I_{p,q} = (\mathrm{id}_p, -\mathrm{id}_q)$, $p + q = n$, and define $\sigma_{p,q}$ the involution on $\mathrm{SL}_n(\mathbb{R})$ obtained by conjugation with $I_{p,q}$. The isotropy group is by definition $\mathrm{SO}_{p,q}(\mathbb{R})$, the group of orientatation perserving isometries of the indefinite form $\sum_{i=1}^{q} x_{i+p}^2 - \sum_{i=1}^{p} x_i^2$ Note that $\mathrm{SO}_{1,1}(\mathbb{R})$ is the diagonal group in $\mathrm{SL}_2(\mathbb{R}) \simeq \mathrm{SO}_{1,2}(\mathbb{R})$. One can also take $G = \mathrm{SO}_{p,q}(\mathbb{R})$, and $\sigma_{p',q'}$ giving rise to some $\mathrm{SO}_{p',q'}(\mathbb{R}) < \mathrm{SO}_{p,q}(\mathbb{R})$. Of particular importance is $\mathrm{SO}_{p,q-1}(\mathbb{R}) < \mathrm{SO}_{p,q}(\mathbb{R})$ from $I_{p+q-1,1}$ since $\mathrm{SO}_{p,q}(\mathbb{R})/\mathrm{SO}_{p,q-1}(\mathbb{R})$ is identified with the hyperboloid

$$\sum_{i=1}^{q} x_{i+p}^2 - \sum_{i=1}^{p} x_i^2 = 1$$

Any involution σ on G induces an involution on $\mathfrak{g}$, which we shall denote by the same letter. Then $\mathfrak{g}$ splits into σ-eigenspaces for the eigenvalues ± 1

$$\mathfrak{g} = \mathfrak{h}_\sigma \oplus \mathfrak{q}_\sigma.$$

In particular $\mathfrak{h}$ is the Lie algebra of H. Note that we have

$$[\mathfrak{h}_\sigma, \mathfrak{h}_\sigma] \subset \mathfrak{h}_\sigma,\ [\mathfrak{h}_\sigma, \mathfrak{q}_\sigma] \subset \mathfrak{q}_\sigma,\ [\mathfrak{q}_\sigma, \mathfrak{q}_\sigma] \subset \mathfrak{h}_\sigma$$

and for any decomposition with such brackets relations there is an involution giving raise to this decomposition.

Example 2.6. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ and $\sigma(X) = -X^T$ inverse transpose. Then the above decomposition is between symmetric and skew-symmetric traceless matrices.

Definition 2.7. We shall write ad_X the map $Y \mapsto [X, Y]$. The killing form $B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ is non-degenerate iff G is semi-simple and negative definite if G is compact. An involution θ is called *Cartan involution* if $B_\theta = -B(X, \theta(Y))$ is symmetric and positive definite. Note that the adjoint of ad_X with respect to this inner product becomes $-\mathrm{ad}_{\theta(X)}$, and thus selfadjoint on $\mathfrak{p}_\theta$

Example 2.8. For $\mathfrak{sl}_n(\mathbb{R})$, $B(X,Y) = 2n\,\mathrm{Tr}(XY)$, so that $B_\theta(X,Y) = -2n\,\mathrm{Tr}(X\theta(Y)) = 2n\,\mathrm{Tr}(XY^T)$. But $\mathrm{Tr}(XY^T)$ is an inner product on the space of $n \times n$ matrices making θ a Cartan involution.

Proposition 2.9.

- B_θ is symmetric and positive definite
- B is invariant under any automorphism
- $\mathfrak{k} \perp \mathfrak{p}$ with respect to both B and B_θ

Definition 2.10. The decomposition $\mathfrak{h}_\theta \oplus \mathfrak{q}_\theta$ for a Cartan involution θ is called a Cartan pair.

Example 2.11. For $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k} = \mathfrak{so}_n(\mathbb{R})$. Since $\mathfrak{p}$ consists of symmetric matrices, any $Y \in \mathfrak{p}$ can be diagonalized, $Y = kZk^{-1} = \mathrm{Ad}_k Z$ for some Z diagonal (and traceless) and $k \in K$. Let $\mathfrak{a} < \mathfrak{g}$ be the diagonal traceless matrices then $\mathfrak{p} = \mathrm{Ad}_K \mathfrak{a}$.

Theorem 2.12. *A Cartan involution is unique up to an inner automorphism, i.e. $\theta = f \circ \theta' \circ f^{-1}$ and $f = \mathrm{Ad}_g$ for some $g \in G$. For any involution σ, there exists a Cartan convolution that commutes with σ.*

Theorem 2.13. *For a Cartan pair $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, K is a maximal compact subgroup of G. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $A = \exp \mathfrak{a}$. Then $G = K \exp \mathfrak{p}$ (in fact $(k, X) \to k\exp(X)$ is a diffeo, $\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}_k \mathfrak{a}$ (in fact for any maximal $\mathfrak{a}, \mathfrak{a}'$ in $\mathfrak{p}$ are K-conjugates and $G = KAK$.*

Proof. Assume G is a classical group, say $G \subset \mathrm{GL}(\mathbb{C}, n)$ and θ is Inverse conjugate transpose. Then there is a unique polar decomposition $g = k\exp(X)$ with k unitary and X Hermitian (exp is surjective on the positive definite Hermitian matrices since it is on diagonal matrices). Now $\bar{k}^T = k^{-1}$, $\bar{X}^T = X$

$$\theta(g) = k \exp -X, \quad \theta(g)^{-1} g = \exp 2X$$

which implies $\exp X \in G$ (using the fact that if $\exp X$ in a algebraic group then X is in the Lie algebra). Since $g \in G$, also $k \in G \cap U(n) = K$ is compact. We see that K must be maximal, since else it contains an element of $\exp \mathfrak{p}$ but any non-trivial element gives an unbounded subgroup.

Given $\mathfrak{a}, \mathfrak{a}'$ take Z, Z' such that no root $\Sigma^{\mathfrak{a}}$ resp. $\Sigma^{\mathfrak{a}'}$ vanishes. Consider the curve

$$K \ni k \mapsto B(\mathrm{Ad}_k Z, Z')$$

. Let $k \in K$ be the minimum (which exists by compactness of K). Its derivative,

$$B(\mathrm{ad}_H \mathrm{Ad}_k Z, Z') = B([\mathrm{Ad}_k Z, Z'], H) = 0$$

for $H \in \mathfrak{k}$ vanishes, but $B(H,H) < 0$ for any $H \in \mathfrak{k}$, and thus $[\mathrm{Ad}_k Z, Z'] = 0$. Since Z' has non-trivial projection to any $\mathfrak{g}_\lambda^{\mathfrak{a}'}$, $\mathrm{Ad}_k Z \in \mathfrak{g}_0^{\mathfrak{a}'}$. Since $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$, $\mathfrak{m} < \mathfrak{k} \perp \mathfrak{p}$ and as $\mathrm{Ad}_k Z \in \mathfrak{p}$, $\mathrm{Ad}_k Z \in \mathfrak{a}'$. By symmetry of the argument, $\mathrm{Ad}_k Z \in \mathfrak{a}'$. Note also $Z_{\mathfrak{g}}(Z) = Z_{\mathfrak{g}}(\mathfrak{a})$ generates the centralizer by construction. But $\mathfrak{a}'$ commutes now with Z implying that $\mathfrak{a}' < \mathrm{Ad}_k \mathfrak{a}$, and by maximality they are equal.

KAK follows from the previous statements. □

Theorem 2.14. *Let σ be an involution of G with affine symmetric group H and giving rise to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Let θ be a commuting Cartan decomposition with symmetry group K giving rise to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ be a maximal abelian subspace, $A = \exp \mathfrak{a}$. Then $G \subset HAK$*

Proof Sketch. Step 1: $(X, Y, k) \mapsto \exp X \exp Y k$ from $(\mathfrak{p} \cap \mathfrak{h}) \times (\mathfrak{p} \cap \mathfrak{q}) \times K$ to G is a local diffeo onto. Diffeo by local dimensions argument. Assuming a decomposition $g = \exp X \exp Y k$ for the moment. Since $\theta(g^{-1}) = \theta(k^{-1} \exp -Y \exp -X) = k^{-1} \exp(Y) \exp(X)$, we have

$$g\theta(g^{-1}) = \exp X \exp 2Y \exp X$$

We already know $G = \exp \mathfrak{p} K$ uniquely, and we want to assume $g = \exp S$ for $S \in \mathfrak{p}$, in particular $\theta(g^{-1}) = g$ and the LHS is $\exp 2S$ Apply σ, to see (σ fixes $\mathfrak{h}$ thus X)

$$\exp 2\sigma(S) = \exp X \exp -2Y \exp X.$$

Combining both gives

$$\exp 2\sigma(S) = \exp 2X \exp -2S \exp 2X$$

and thus

$$\exp -S \exp 2\sigma(S) \exp -S = (\exp -S \exp 2X \exp -S)^2$$

which we may rewrite as

$$\exp 2X = \exp S \exp T \exp S$$

with

$$\exp 2T = \exp -S \exp 2\sigma(S) \exp -S$$

These formulas show that X and Y are uniquely determined, and how to construct them given g.

We reduce to show

Step 2: $\exp \mathfrak{p} \cap \mathfrak{q} \subset HAK$.

Define $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q})$. By the bracket relations of involution, it is a sub lie algebra. Since σ and θ commute, θ preserves the eigenspace decomposition with respect to σ, and thus preserves $\mathfrak{g}_0$ but also the decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ (σ acts by ± 1, so any intersection of an eigenspace is preserved). The associated Lie group G_0 is by definition reductive, and again allows a $K_0 A_0 K_0$ decomposition where $A_0 = A$ and $K_0 = H \cap K$. We now conclude that $\exp \mathfrak{p} \cap \mathfrak{q} \subset G_0 \subset HA_0K$. □

The maps ad_Z for $Z \in \mathfrak{a}$ are commuting, and as remarked before, selfadjoint with respect to B_θ. Introduce the dual $\mathfrak{a}^*$ and for $\lambda \in \mathfrak{a}^*$,

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \mathrm{ad}_Z(X) = \lambda(Z)X \text{ for all } Z \in \mathfrak{a}\}$$

Let Σ consists of all $\lambda \neq 0$ with $\mathfrak{g}_\lambda$, the set of restricted roots. Having chosen a basis on $\mathfrak{a}^*$, one might introduce an ordering on Σ let Σ^+ be the posititve restricted roots. A root in Σ^+ is called simple if it cannot be written as sum as any other two. Remark: Given a basis of $\mathfrak{a}^*$ coming from elements of Σ, then these are simple with respect to some choice of Σ^+ if any other root in Σ can be expressed in either all positive or all negative integer coefficients.

Example 2.15. Let E_{ij} be the elementary matrices in $\mathfrak{sl}_n(\mathbb{R})$ and $Z = \mathrm{diag}(h_1, \ldots, h_n) \in \mathfrak{a}$ then $\mathrm{ad}_Z(E_{ij}) = (h_i - h_j)E_{ij}$. Let $e_j \in \mathfrak{a}^*$ by $e_j(H) = h_j$, then $e_i - e_j$ are precisely such λ for which $\mathfrak{g}_\lambda \neq 0$ forming Σ. Taking the order induced from $e_1, \ldots, e_n$, a root is positive if the first coefficient is positive in that basis (so that $e_1 - e_n$ is the largest posititve root and $e_{n-1} - e_n$ the smallest), and $e_i - e_{i+1}$ form a base of simple positive roots.

Theorem 2.16.

- $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda$ *(orthogonal sum)*
- $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$
- $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ *and hence* $\lambda \in \Sigma$ *implies* $-\lambda \in \Sigma$. *Same for* σ.
- $\mathfrak{g}_\lambda \perp \mathfrak{g}_\mu$ *with respect to* B_θ

We study now the Lie subalgebra of $\mathfrak{g}$,

$$\mathfrak{n} = \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$$

Theorem 2.17. *Assume for the moment that $\sigma = \theta$. Then the above theorem can be extended to say*

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$$

and the Iwasawa decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

and $K \times A \times N \to G$ is a diffeo onto.

Proof. Any $X \in \mathfrak{l}$ has non-zero projection to $\mathfrak{m}$ or $\sum_{\Sigma^+} \mathfrak{g}_{-\lambda}$ together with $\mathfrak{g} = \mathfrak{n} + \mathfrak{g}_0 + \bar{\mathfrak{n}}$ making $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ a direct sum. It is everything since

$$\mathfrak{a} + \mathfrak{m} + (\mathfrak{n} + \bar{\mathfrak{n}}) \ni Z + X_0 + \sum X_\lambda = (X_0 + \sum (X_{-\lambda} + \theta X_{-\lambda}) + Z + \sum (X_\lambda - \theta X_{-\lambda}) \in \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

For the group level one uses that if $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t}$ of two subalgebras then the differential of the multiplication map vanishes nowhere. The image is closed since K is compact and AN are closed (for any subsequence, take a subseqeuence where the K part converges, then take limit in AN, still of product form). The image is also open. Thus everything. Now also multiplication from $A \times N$ to AN is smooth and onto. □

Definition 2.18. The hyperplanes in $\mathfrak{a} \simeq \mathfrak{a}^*$ defined by $\ker \lambda$ cut $\mathfrak{a}$ into finitely many open regions $\{\mathcal{C}\}$ called *Weyl chambers*. For any set of simple roots $\Delta \subset \Sigma$ there is a unique $\mathcal{C}_\Delta$ defined by the intersection of the half-spaces $\lambda > 0$ in $\mathfrak{a}$ where $\lambda \in \Delta$, and Σ_Δ^+ denotes the positive roots with respect to Δ, i.e. those λ for which $\lambda(\mathcal{W}_\Delta) > 0$. Denote by $\mathfrak{n}_\Delta = \sum_{\lambda \in \Sigma_\Delta^+} \mathfrak{g}_\lambda$ and

$$N_\Delta =< \exp \mathfrak{n}_\Delta >, \quad A_\Delta = \exp \overline{\mathcal{C}_\Delta}$$

Any Weyl chamber contains exactly one root, the maximal element with respect to the ordering.

Example 2.19. Picture of triangulation of equilateral triangles coming from A_2. If α, β are two simple roots $\alpha + \beta$ is maximal and contained in the cone of the corresponding Weyl chamber. It is the highest weight of the adjoint representation.

Proposition 2.20. There exists a a shrinking family of open neighborhoods N_ϵ of $e \in N_\Delta$ invariant under conjugation by A_Δ, i.e. for any open $e \in U$ there is $V_\epsilon \subset O$ with

$$e \in a^{-1} V_\epsilon a \subset V_\epsilon \subset U$$

for any $a \in A_\Delta$

Proof. Let $X = \sum_{\lambda \in \Sigma^+} x_\lambda X_\lambda \in \mathfrak{n}$ where X_λ spans the one-dimensional space $\mathfrak{g}_\lambda$. Let $c_a : N \to N$ the conjugation map $n \mapsto ana^{-1}$, its derivative acts on $\mathfrak{n}$ by $\mathrm{Ad}(a) : \mathfrak{n} \to \mathfrak{n}$ which is related the previous adjoint action by $\mathrm{Ad}(\exp Z) = \exp(\mathrm{ad}_Z)$, and so $\mathrm{Ad}(a^{-1})X_\lambda = \exp(-\lambda(Z))$ for $a = \exp Z \in A_\Delta$,

$$\mathrm{Ad}(a^{-1})X = \sum_{\lambda \in \Sigma^+} x_\lambda \exp(-\lambda(Z)) X_\lambda \in \mathfrak{n}$$

and we see that a^{-1} contracts as $\lambda(Z) > 0$. Take V_ϵ to be a product neighbourhood.

□

Theorem 2.21. *Let $M = Z_K(A)$, then $H \times M \times A \times N \to G$ is open in a neighborhood of the identity in G.*

Proof. It suffices to show $\mathfrak{h} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n} = \mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{g}_0 \oplus \bar{\mathfrak{n}}$. We decompose any X with respect to that decomposition and thus assume $X \in \bar{\mathfrak{n}} \oplus \mathfrak{g}_0$. For the $\bar{\mathfrak{n}}$ part we observe that also $\sigma(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$ since

$$[Z, \sigma(X)] = \sigma([\sigma(Z), X] = -\sigma([Z, X] = -\lambda(Z)\sigma(X)$$

for $X \in \mathfrak{g}_\lambda$ and $X + \sigma(X) \in \mathfrak{h}$.

Thus for any $X \in \bar{\mathfrak{n}} = \oplus_{\lambda \in \Sigma^+} \mathfrak{g}_{-\lambda}$,

$$X = (X + \sigma(X)) - \sigma(X) \in \mathfrak{h} \oplus \mathfrak{n}.$$

It remains to show $\mathfrak{g}_0 \subset \mathfrak{m} + \mathfrak{a} + \mathfrak{h}$.

Remark: If $\sigma = \theta$, i.e. $\mathfrak{a}$ maximal in $\mathfrak{p}$, we have $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ (orthogonal sum) where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. Since $\mathfrak{a}$ in general smaller, g_0 is larger and the claim is that the new contribution is along $\mathfrak{h}$.

Both θ and σ preserve $\mathfrak{g}_0$ by the same calculation we just did, in particlar we have a direct sum decomposition of $\mathfrak{g}_0$ (given by $2X = X + \sigma(X) + X - \sigma(X)$ and in particular both parts are in $\mathfrak{g}_0$). For θ we have in fact $\mathfrak{k} \oplus \mathfrak{p}$ with respect to B_θ giving

$$\mathfrak{g}_0 = \mathfrak{k} \cap \mathfrak{g}_0 \oplus \mathfrak{p} \cap \mathfrak{g}_0$$

which respects θ.

We see that by definition of $\mathfrak{m}$, $\mathfrak{k} \cap \mathfrak{g}_0 = \mathfrak{m}$. ($\mathfrak{k} \cap \mathfrak{g}_0$ consists of the kernel of $\mathrm{ad}_\mathfrak{a}$ contained in $\mathfrak{k}$.)

Now we also decompose $\mathfrak{p} \cap \mathfrak{g}_0 = \mathfrak{p} \cap \mathfrak{h} \cap \mathfrak{g}_0 + \mathfrak{p} \cap \mathfrak{q} \cap \mathfrak{g}_0$ and $\mathfrak{p} \cap \mathfrak{g}_0 < \mathfrak{h} + \mathfrak{a}$ follows if $\mathfrak{p} \cap \mathfrak{q} \cap \mathfrak{g}_0 < \mathfrak{a}$. But any $X \in \mathfrak{g}_0$ commutes with $\mathfrak{a}$, which as chosen maximal abeliean in $\mathfrak{p} \cap \mathfrak{q}$, in particular contains $\mathfrak{p} \cap \mathfrak{q} \cap \mathfrak{g}_0$. □

3 Wavefront Lemma

Theorem 3.1. *For any open neighbourhood U of $e \in G$ there is $V \subset G$ open such that*

$$HVg \subset HgU$$

for all $g \in AK$.

Proof. Assume first that $g \in A$. Then $g \in \exp(\bar{\mathcal{C}})$ for some Weyl chamber. Let N be the corresponding unipotent subgroup, with a contraction invariant neighborhoods V_N. We also let V_M, V_A, neighbourhoods in M and A and put $V = HV_MV_AV_N$ a neighbourhood of G by $HMAN$ decomposition, by which we may also assume that $V_MV_AV_N \subset U$

$$HVg = HV_MV_AV_Ng = HgV_MV_A(g^{-1}V_Ng) \subset HgV_MV_AV_N \subset HgU$$

This $V = V_{\mathcal{C}}$ depends on the Weyl chamber, and we take the intersection of all of them.

For general $g = ak$, we may choose that $U' \subset U$ which is K-conjugation invariant and take V coming the above construction for a. Then

$$HVg = HVak \subset HaU'k = Hakk^{-1}U'k = Hgk^{-1}U'k \subset HgU$$

□

4 Equidistribution

Let $\Gamma < G$ be a lattice and let $X = \Gamma\backslash G$. We assume that Γ projects densely onto G/G' for any G' normal noncompact Liegroup $G' \subset G$. This implies that $L^2(X)$ does not contain non-trivial G_i-invariant vectors for any i, and therefore, by Howe-Moore,

Theorem 4.1. *The action of G on X is mixing, that is for any $\alpha, \beta \in L^2(X)$,*

$$\int_X \alpha(xg)\beta(x)dx \to \frac{1}{m(X)} \int_X \alpha \int_X \beta$$

Assume that H is such that $\Gamma \cap H$ intersects H in a lattice. Then ΓH is a closed orbit of finite volume, naturally identified with $\Gamma \cap H\backslash H$ of measure $m(Y)$ induced by a fixed Haar measure on H. We may push these measures to measures on ΓHg.

Theorem 4.2. *The translates Yg, $Y = \Gamma H$ become equidistributed in X as $Hg \to \infty$ in H/G:*

$$\frac{1}{m(Y)} \int_{Yg} \alpha(y)dy \to \frac{1}{m(X)} \int_X \alpha(x)dx.$$

for any $\alpha \in C_c(X)$.

Proof. Let $Hg_n \to \infty$ in $H\backslash G$, $g_n \in AK$. Let (U, ϵ) such that $\alpha(gu)$ is ϵ-close to $\alpha(g)$ fpr all $u \in U$. By the wave front lemma, there is $HVg \subset HgU$ for all g in AK and by mixing,

$$\frac{1}{m(YV)} \int_{YVg_n} \alpha(g)dg = \frac{1}{m(YV)} \int_{\Gamma\backslash G} \chi_{YV}(g)\alpha(gg_n)dg \to \frac{1}{m(X)} \int_X \alpha(g)dg.$$

The LHS is a convex combination of the integrals

$$\frac{1}{m(Y)} \int_{Yg_nu} \alpha(h)dh$$

which are ϵ-close to $\frac{1}{m(Y)} \int_{Yg_n} \alpha(h)dh$. □

5 Counting

Theorem 5.1.

$$|\{M \in \mathrm{Mat}_{dd}(\mathbb{Z}) | \det M = a, \|M\| \leq R\}| \asymp c_a R^{d(d-1)}$$

$V = \{M \in \mathrm{Mat}_{dd}(\mathbb{R}) | \det M = a\} = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) / \mathrm{SL}_2(\mathbb{R})$. Claim: $V(\mathbb{Z})$ finite union of $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$-orbits. Action of $G \times G$ on V by gMh^{-1}. $H = \Delta G$. The maximal abelian space $\mathfrak{a}$ is $A' = \{(a, a^{-1})\} \in A \times A$, and

$$G \times G = (K \times K)A'H$$

Theorem 5.2. *V_a level set of the standard quadratic surface of signature (m, n), $a \in \mathbb{Z}$ and assume $V(\mathbb{Z})$ not empty then*

$$|V(Z) \cap B_R^V| \asymp c_a R^{m+n-2}$$