VEECH: COUNTING WITH RATES

Abstract. A counting function N(T,g) can be approximated in both parameters T and g. We survey all possible combinations of these smoothing arguments.

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1. Counting point sets in the plane - Introduction

1.1. **Approximative Counting.** We consider the standard left action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2\backslash 0$ and let $\Gamma < G = \mathrm{SL}_2(\mathbb{R})$ be a non-uniform lattice and $v \in \mathbb{R}^2 - \{0\}$ such that Γv is a discrete set in $\mathbb{R}^2$. Γ to be a lattice means that the induced Haar measure $m_{G/\Gamma}$ on G/Γ is finite. Non-uniformity of Γ is equivalent to Γ containing a non-trivial unipotent element. Γv is discrete if and only if v is stabilized by a unipotent subgroup Γ_∞ in Γ. The first example are the primitive points $\mathrm{SL}_2(\mathbb{Z})e_1$ in $\mathbb{Z}^2$. We are concerned with the counting problem of

$$\Gamma v \cap B \tag{1}$$

for a set $B \subset \mathbb{R}^2$. Identify $\mathbb{R}^2\backslash 0 = G/N$ and $G^\infty = G/\Gamma_\infty$. Then G^∞ covers both G/Γ and $\mathbb{R}^2\backslash 0$. Define the Siegel-Veech transform taking a function f on G/N to functions on G/Γ,

$$\Theta_f(g\Gamma) = \sum_{\gamma \in \Gamma/\Gamma_\infty} f(g\gamma N).$$

Note that $g\Gamma v \cap B = \Theta_{\mathbb{1}_B}(g\Gamma)$ and the classical strategy is to obtain asymptotics of 1 by an average procedure of $\Theta_{\mathbb{1}_B}(g\Gamma)$, see Section 2 DRS which we follow. We define an adjoint of Θ_f as follows: For F on G/Γ we let F° on G/N by

$$F^\circ(gN) = \int_{N/\Gamma_\infty} F(gn\Gamma)dn.$$

Then

$$\langle \Theta_f, F\rangle_{G/\Gamma} = \int_{G^\infty} f(gN)F(g\Gamma_\infty)dg = \langle f, F^\circ\rangle_{G/N}.$$

Suppose now that $f = \beta_t \in L^1(G/N)$ such that $\beta_t \to m_{G/N}$. Identifying G/N measurably with $\mathbb{R}^2$ we are thus asking β_t to be a Foelner sequence, say euclidean balls B_t of radius t. Let $F = F_\varepsilon$ be a dirac

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approximation of $\Gamma \in G/\Gamma$.[1] We additional assume the support of β_t to be *wellrounded*, which for now means

$$\Theta_{\beta_t}(g_1\Gamma) \le \Theta_{\beta_{t\rho(\varepsilon))}}(g_2\Gamma), \tag{2}$$

for any $g_1, g_2 \in B^G_\varepsilon$ and $\rho(\varepsilon) \to 1$ as $\varepsilon \to 0$ and that $\lim_t \frac{m_{G/N}(B_t)}{m_{G/N}(B_{\rho(\varepsilon)t})} \to 1^2$ as $\varepsilon \to 0$.

This allows to upgrade the weak* limit of $\beta_t \to m_{G/N}$,

$$\langle \beta_t, F^\circ_\varepsilon\rangle_{G/N} \to m_{G/N}(F_\varepsilon) \text{ as } t \to \infty$$

to the pointwise statement

$$\lim_{t\to\infty} \Theta_{\beta_t}(\Gamma)/|\beta_t| = 1$$

see Lemma 2.3 DRS. Clearly, this approach can be made qualitative if given sufficient information on $\beta_t \to m_{G/N}$. Sarnak's theorem on low lying horocycles reads

Theorem 1.1. *For any smooth $F \in L^2(G/\Gamma)$ and any $g = kan$, $a = \operatorname{diag}\left(y^{-1}, y\right)$*

$$F^\circ(gN) = m_{G/\Gamma}(F) + \mathcal{O}(\mathcal{S}(F)y)$$

as $y \to 0$.

Let $E(gN)$ denote the error term as function on G/N, so that

$$\langle\Theta_{\beta_t}, F\rangle_{G/\Gamma} = |\beta_t| m_{G/\Gamma}(F) + \beta_t \star E \tag{3}$$

where the error term is a simple integral over $\mathbb{R}^2$. A careful analysis of continuity of $\Theta_{\beta_t}(g\Gamma)$ in both variables t and g in the sense of equation 2 will allow us to count the intersection of Γv in dilations of general domains $B_t = t\Omega$. We actually see that equation 3 is strong enough to count in any domain, only given that $\langle\Theta_{\beta_t}, F\rangle_{G/\Gamma}$ is close to $\Theta_{\beta_t}(\Gamma)$.

Remark 1.2. Note that Theorem 1.1 is actually somewhat overkill of what we need - decay of $\delta_g \star F^\circ$ as $g \to \infty$ compared to $\beta_t \star F^\circ$ as $m_{G/N}(\operatorname{Supp}(\beta_t)) \to \infty$. Discuss relation to [Rogers, Modified' Siegels formula] where it is proven that β_t balls and $F = \Theta_\phi$ itself a Siegel transform over *any* pointset with spherical density d [3], $\beta_t \star F \to m_{\mathbb{R}^2}(F)$.

1.2. **Perron Formula and Sobolev Spaces.** Let us introduce now the spectral view point and restrict to K-invariant functions f so that Θ_f can be put on $\mathbb{H}/\Gamma$ where we consider the right action of G on $\mathbb{H}$ by Mobius transformation, i.e. $g.z = g^{-1}z$ and identify $\mathbb{H} = K\backslash G$. The spectral theory of $L^2(\mathbb{H}/K)$ begins with a Perron formula

$$\Theta_f(z) = \frac{1}{2\pi i}\int_{\operatorname{Re} s=2} \hat{f}(-s)E(z,s)ds \tag{4}$$

where $\hat{f}$ is the Mellin transform and the classical Eisenstein series $E(z,s) = \sum_{\gamma\in\Gamma/\Gamma_\infty} y(\gamma^{-1}z)^s$. Indeed, the so called continuous spectrum is by definition the $L^2(K\backslash G/\Gamma)$ closure of all functions of the form Θ_f where f are smooth and compactly supported.

The spectral theorem rests on the meromorphic continuation of E and their asymptotics for $\operatorname{Im}(s) \to \infty$. For smooth f one is allowed to apply a Cauchy-Residue argument to move the integral to the critical line $\frac{1}{2}$. Passing the point $s = 1$, a residue $m_{G/\Gamma}(\Theta_f)$ appears which by Siegels formula equals $m_{G/N}(f)$. It remains to show that the continuous wavepacket is neglable. This requires

Theorem 1.3.

$$\int_{-T}^{T} |E(z, \tfrac{1}{2} + it)|^2 dt = \mathcal{O}(T^2).$$

It is also the driving spectral bound for Theorem 1.1, and moreover, can be used to proof Weyl's law (of the continous spectrum) on $\mathbb{H}/\Gamma$. Theorem 1.3 turns out to be a consequence of the (continuous) spectral theorem and Bessel's inqueality. We proof it later (13.1).

One may only use the asymptotitic behaviour for $\operatorname{Re} s > 1/2$ for which the decomposition coming from equation 4 will make sense also for (non-smooth) characteristic function $\mathbb{1}_{B_t}$ and thus leads to

[1]Note that F°_ε is a smoothening of $\delta_{\Gamma/N} = \sum_{\Gamma/\Gamma_\infty} \delta_{\gamma N}$.

[2]more precisely formlated with lim inf-sup

[3](i.e. $N_T/T^2 \to d$)

an asymptotic count without any further approximation argument. One might use the final spectral decomposition

$$\Theta_f(z) = \operatorname{Res}_{s=1} E(z,s) \int \Theta_f(z)dz + \frac{1}{4\pi i} \int_{\operatorname{Re} s = \frac{1}{2}} \langle \Theta_f(\cdot), E(\cdot, s)\rangle E(z,s)ds$$

but this is only understood in the L^2 sense for f not smooth, and we better again do a smoothing argument that requires corresponding approximations as in 2 but for deformations along A only.

The spectral decomposition for general f, not just radial, allows therefore the treatment for general shapes given that one can bound $\langle \Theta_f(\cdot), E_m(\cdot, s)\rangle$. In general, this is best approached by a decomposition of a general function f into K-eigenvectors, where one keeps track on dependency along A, and is conveniently expressed in terms of Sobolev norms. See Bernstein-Reznikov for a treatment, and Venkatesh, Einsiedler-Margulis-Venkatesh, Kelmer-Kontorovich, Sarnak-Ubis for recent uses and Flaminio-Forni for an indepedendent and selfcontained discussion.

2. Counting using the spectral Theory of Eisenstein Series

2.1. **Eisenstein series I - Veech and Counting.**

Standing assumption are that we count at the infinity cusp and the $\mathrm{SL}_2(\mathbb{R})$-action on $L^2(G/\Gamma)$ is tempered. The first assumption allows us to use the dual picture between counting $g\Gamma v$ in $B^{\mathbb{R}^2}(0,R)$ for $v = e_1$ and $\Gamma g^{-1} i$ in $\mathcal{H}(R) = [-1/2, 1/2] \times [1/R^2; \infty) \subset \mathbb{H}$ since $(\Im g^{-1} i)^s = \|gv\|^{-2s}$, see (16.9) and the discussion after in Veech[Siegel measures]. The stabilizer in Γ is denoted by Γ_∞. The second assumption implies lack of non-trivial poles of the Eisenstein series in the critical strip. Let

$$N(g,R) = |g\Gamma v \cap B_R|.$$

Let us recall more notation from Chapter 16 Veech[Siegel measures]. Define the Eisenstein series as

$$E(z,s) = \sum_{\gamma \in \Gamma/\Gamma_\infty} \|g\gamma v_0\|^{-2s} \tag{5}$$

in the variable $z = g^{-1} i$. With a Lebesgue-Stielties integral representation one is lead to

$$E(z,s) = \int_0^\infty \frac{dN(g,R)}{R^{2s}} = 2s \int_0^\infty \frac{N(g,R)}{R^2} R^{1-2s} dR \tag{6}$$

valid for any $\Re(s) > 1$. Veech continues to calculate the residual limit of $E(z,s)$ at $s = 1$ without further knowledge assumed on $E(z,s)$ but using the main term count of $N(g,R)$. Indeed, the right hand side of equation 6 is absolute convergent using that $N(g,R) = \mathcal{O}(R^2)$. We shall use a smooth version of formula 6, see Theorem 2.3.

Theorem 2.1 (Veech,Siegel measures).

$$\lim_{s\to 1} (s-1)E(z,s) = c(\Gamma, v)\pi$$

where s in the left hand side is restricted to lie in some explicit set $U(\sigma)$ and the right hand satisfies

$$N(g,R) = c(\Gamma, v)\pi R^2 + o(R^2)$$

We see that $c(\Gamma, v)$ is the Siegel-Veech constant. Also recall the following theorem of Veech.

Theorem 2.2 (Theorem 14.11, Veech, Siegel measures). *For any $\phi \in C_c(\mathbb{R}^2)$, denote by $T\phi$ the map $x \mapsto \phi(x/T)$.*

$$\lim_{T\to\infty} \frac{1}{T} \Theta_{T\phi} = c(\Gamma, v) \int_{\mathbb{R}^2} \phi$$

where Θ_f is the Siegel-Veech transform.

Also introduce the Mellin transform of ψ,

$$\hat{\psi}(s) = \int_0^\infty \psi(y) y^{s-1} dy.$$

From the integral representation in equation 6 we see $\frac{E(z,s)}{2s} = \widehat{N}(-2s)$ and by Mellin-inversion

$$N(R) = \frac{1}{2\pi i} \int_{(c)} R^{-s} \widehat{N}(s) = \frac{1}{2\pi i} \int_{(c)} R^{2s} \widehat{N}(-2s) 2ds = \frac{1}{2\pi i} \int_{(c)} \frac{E(z,s)}{s} R^{2s} ds \tag{7}$$

for any $\operatorname{Re} c > 1$ so that $E(z,s)$ is absolute convergent.

For $f : \mathbb{R}^2 \to \mathbb{R}$, the function $\Theta_f(g\Gamma) = \sum_{\gamma\in\Gamma/\Gamma_\infty} f(g\gamma v_0)$ is known as *Siegel-Veech transform* in the flatsurface community. For $f = \mathbb{1}_{B(0,R)}$, this function is $\mathrm{SO}_2(\mathbb{R})$-invariant, and one can equivalently write it as *incomplete theta series*, which takes $\psi : \mathbb{R}_{>0} \to \mathbb{R}$ to

$$E(\psi, z) = \sum_{\gamma\in\Gamma/\Gamma_\infty} \psi(\mathrm{Im}(\gamma z))).$$

We note that $E(z, s) = E(\psi, z)$ for $\psi = y^s$ if we were to allow such functions.

The associated (spherical) Eisenstein series $E(z, s)$ can be meromorphically extended to the whole complex plain, and satisfies a functional equation (see e.g. Theorem 6.5 Iwaniec)

$$E(z, s) = \phi(s)E(z, 1 - s) \tag{8}$$

The incomplete theta series enjoys the following spectral decomposition Terras[Lemma 3.7.1] for $\mathrm{SL}_2(\mathbb{Z})$ and see Borel for general Γ. It is essentially the spectral Theorem for the continuous part of $L^2(\Gamma\backslash\mathbb{H})$ with respect to the hyperbolic Laplacien. [4]

Theorem 2.3. *Let $\psi : \mathbb{R}_{>0} \to \mathbb{R}$ be smooth and compactly supported.*

$$E(\psi, z) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=c} \hat{\psi}(-s)E(z, s)ds \text{ for any } c > 1. \tag{9}$$

$$\langle E(\psi, \cdot), E(\cdot, s)\rangle = \hat{\psi}(\bar{s} - 1) + \phi(\bar{s})\hat{\psi}(-\bar{s}) \tag{10}$$

$$E(\psi, z) = \mathrm{Res}_{s=1}\, E(z, s)\int E(\psi, z)dz + \frac{1}{4\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} \langle E(\psi, \cdot), E(\cdot, s)\rangle E(z, s)ds \tag{11}$$

We note that $\int E(\psi, z)dz = \hat{\psi}(-1)$ and $\int_{\mathrm{Re}\, s=\frac{1}{2}} \langle E(\psi, \cdot), E(\cdot, s)\rangle E(z, s)ds = 2\int_{\mathrm{Re}\, s=\frac{1}{2}} \hat{\psi}(-s)E(z, s)ds$ and all integrals are absolute convergent.

Let us explain how equation 11 implies the the following polynomial error rate in counting. Since we wish to relate the size of the integral on the right hand side of equation 11 with smoothness of ψ, we shall also include a proof of equation 9,11. The Siegel-Veech transform enjoys the same decomposition, see the proof of Proposition 12.3. This will follow from a spectral decomposition with respect to the angular coordinates on $\mathbb{R}^2$ (corresponding to the action of $K = \mathrm{SO}_2(\mathbb{R})$), and requires twisted (non-spherical) Eisenstein series, where the twist happens precisely over the K-characters. We will introduce them in context for counting in sectors in Section 2.2.

Theorem 2.4 (Iwaniec Proposition 7.2)**.**

$$\int_{-T}^{T} |E(g, \tfrac{1}{2} + it)|^2 dt = \mathcal{O}(T^2).$$

Theorem 2.5.

$$N(R, g) = c(\Gamma, v)\pi R^2 + O(R^{4/3+}).$$

Proof. Let $\psi_U : \mathbb{R}_+ \to \mathbb{R}$ smooth

$$\psi_U(t) = \begin{cases} 1, & \text{for } t \leq 1 - 1/U \\ 0, & \text{for } t \geq 1 + 1/U \end{cases}$$

and consider also its Mellin transform.

$$\Psi_U(s) = \int_0^\infty \psi_U(y)y^{s-1}dy$$

We shall use this to approximate $N(g, R)$ by the incomplete Eisenstein transform of two such $\phi_U^- \leq \mathbb{1}_{(0,1]} \leq \phi_U^+$. Write $N_T(g) = N(g, R)$.

$$\sum_{\Gamma_\infty\backslash\Gamma} \phi_U^-(y(\gamma g)^{-1}T^{-1}) \leq N_T(g) \leq \sum_{\Gamma_\infty\backslash\Gamma} \phi_U^+(y(\gamma g)^{-1}T^{-1}) \tag{12}$$

For $\phi_U = \phi_U^\pm$, we have by 11 and multiplicative property of the Mellin transform

$$\sum_{\Gamma_\infty\backslash\Gamma} \phi_U(y(\gamma g)^{-1}T^{-1}) = \mathrm{Res}_{s=1}\left(E(g, s)\Psi_U(s)T^s\right) + \frac{1}{2\pi i}\int_{(\frac{1}{2})} E(g, s)\Psi_U(s)T^s.$$

[4]...and also decide on the various definitions involving the choice of s or $-s$ and $\frac{1}{2}$ normalization factors

Since the integral on the right hand side is absolute convergent, it is asymptotically $\mathcal{O}(T^{1/2})$. We still have to explicate the depedency on U however.

Let us start with the the following simple lemma.

Lemma 2.6.

$$\Psi_U(s) = \begin{cases} s^{-1} + \mathcal{O}(U^{-1}), & \text{as } U \to \infty \\ \mathcal{O}\left(\frac{1}{|s|}\left(\frac{U}{1+|s|}\right)^k\right), & \text{as } |s| \to \infty \end{cases}$$

for any $k > 0$.

We refer to the Appendix for a proof.

For $\phi_U = \phi_U^{\pm}$, by Mellin inversion and integration by parts in the Lebesque-Stieltjes sense (the right hand site is absolutely convergent for any vertical line > 1)

$$\sum_{\Gamma_\infty \backslash \Gamma} \phi_U(y(\gamma g)^{-1}T^{-1}) = \frac{1}{2\pi i}\int_{(1+)} E(g,s)\Psi_U(s)T^s ds.$$

This recovers equation 9.

We could proof therefore a form of equation 11 when applying Cauchy's residue theorem to sthe trip between 1/2 and 2. For this, we are reuqired to have "reasonable" growth of the Eisenstein series, which together with super-polynomial decay of Ψ_U from Lemma 9.2 justies the shift of contour by the Phragmén–Lindelöf principle (we will return to this in section 5):

$$= \mathrm{Res}_{s=1}\left(E(g,s)\Psi_U(s)T^s\right) + \frac{1}{2\pi i}\int_{(\frac{1}{2})} E(g,s)\Psi_U(s)T^s ds$$

$$= \frac{1}{Vol(G/\Gamma)}\left(T + \mathcal{O}(T/U)\right) + \frac{1}{2\pi i}\int_{(\frac{1}{2})} E(g,s)\Psi_U(s)T^s ds \tag{13}$$

Let us move on to our initial goal to bound

$$\int_{(\frac{1}{2})} E(g,s)\Psi_U(s)T^s ds \tag{14}$$

with concrete dependence on U.

Restricting our attention to the upper half, we have (using the second formula of Lemma 9.2) that

$$\left|\int_0^\infty E(g,\tfrac{1}{2}+it)\Psi_U(\tfrac{1}{2}+it)T^{\frac{1}{2}+it}dt\right| \ll T^{\frac{1}{2}}\int_0^\infty |E(g,\tfrac{1}{2}+it)|\frac{U^\alpha}{1+|t|^{1+\alpha}}dt.$$

Cauchy-Schwarz to $\frac{|E(g,\frac{1}{2}+it)|}{1+|t|^\beta} \cdot \frac{1+|t|^\beta}{1+|t|^{1+\alpha}}$,

$$\ll T^{\frac{1}{2}}U^\alpha\sqrt{\int_0^\infty |E(g,\tfrac{1}{2}+it)|^2\frac{1}{1+|t|^{2\beta}}dt}.\sqrt{\int_0^\infty\left(\frac{1+|t|^\beta}{1+|t|^{1+\alpha}}\right)^2 dt}.$$

The last term is finite if $\beta - 1 - \alpha < -\frac{1}{2}$.

For the first term, define $F(T) = \int_0^T |E(g,\frac{1}{2}+it)|^2 dt = \mathcal{O}(T^2)$, then

$$\int_0^\infty |E(g,\tfrac{1}{2}+it)|^2\frac{1}{1+|t|^{2\beta}}dt = \frac{F(t)}{1+|t|^{2\beta}}\Big|_0^\infty - \int_0^\infty F(t)\mathcal{O}(\frac{1}{1+|t|^{2\beta+1}})\Big)\, dt.$$

The first summand is finite for $\beta > 1$, the second for $2 - 2\beta - 1 < -1$, which again reduces to $\beta > 1$. A good choice for α therefore is

$$\alpha > \frac{1}{2}.$$

This leads to

$$\int_{(\frac{1}{2})} E(g,s)\Psi_U(s)T^s ds = \mathcal{O}(T^{\frac{1}{2}}U^{\frac{1}{2}+}) \tag{15}$$

which should match the smoothing error $\frac{T}{U}$, i.e. $T^{1/2} = U^{3/2+}$ accumulating to the error term

$$\mathcal{O}(T^{2/3+}) = \mathcal{O}(R^{4/3+})$$

□

2.2. **Eisenstein series II (non-spherical) - Counting in Sectors with Erdos-Turan.** This argument easily upgrades to count in sectors, if given similar asymptotics of Theorem 2.4 for the even weight Eisenstein series

$$E_m(g,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y(\gamma z)^s e\left(m(\theta + \arg(cz+d))\right) \tag{16}$$

where $g = n(x)a(y)k(\theta)$, $z = gi$, $y(z) = \mathrm{Im}(z)$. We also introduce $j_\gamma(g) = e\left((\theta + \arg(cz+d)\right)$.

Indeed, given $N_T(g)$, one obtains a measure μ_T on the circle by restricting to sectors in the ball of volume T, and as such has a Fourier transform,

$$N_{T,m}(g) = \int e(-m\theta)\mu_T(d\theta).$$

which has now integral representation like that in equation 7 with E replaced by E_m.

Theorem 2.7 (Marklof-Strombergssen, Kronecker on Horocycles)**.**

$$\int_{-T}^{T} |E_m(g, \tfrac{1}{2}+it)|^2 dt = \mathcal{O}((T+|m|)^2).$$

Theorem 2.8. *Let $N(R,I,g)$ restrict the counting function to any sector I.*

$$N(R,I,g) = c(\Gamma,v)\frac{|I|}{2}R^2 + O(R^{8/5+}).$$

Proof. We use again the notation $N(R,I,g) = N_{T,I}(g)$. By the lack of poles of E_m (see discussion in Sarnak (Horocycles) or MS), and the above argument,

$$\sum_{\Gamma_\infty\backslash\Gamma} j_\gamma(g)^m \phi_U^{\pm}(y(\gamma g)^{-1}T^{-1}) = \mathcal{O}(T^{1/2}U^{1/2+}m)$$

From the $m=0$ estimate,

$$\left|\sum_{\Gamma_\infty\backslash\Gamma} j_\gamma(g)^m \phi_U^{\pm}(y(\gamma g)^{-1}T^{-1}) - \sum_{\Gamma_\infty\backslash\Gamma : |y(\gamma g)|\geq T^{-1}} j_\gamma(g)^m\right| \leq |N_{T(1\pm\frac{1}{U})}(g) - N_T(g)| = \mathcal{O}(T/U + T^{1/2}U^{1/2+}).$$

so that the m'ths Fourier coefficient of N_T is of order

$$N_{T,m}(g) = \sum_{\Gamma_\infty\backslash\Gamma : |y(\gamma g)|\geq T^{-1}} j_\gamma(g)^m = \mathcal{O}(T/U + T^{1/2}U^{1/2+}m)$$

Erdos-Turan,

$$N_{T,I}(g) = c|I|T + \mathcal{O}(T^{2/3+} + T/M + \sum_{m\leq M}\frac{1}{m}|N_{T,m}|) = c|I|T + \mathcal{O}(T/M + (\log M)T/U + MT^{1/2}U^{1/2+})$$

Set $M = T^{1/5}, U = T^{1/5}$ to get $\mathcal{O}(T^{4/5+})$. □

3. Wellroundedness, Average Counting, Horocycles

This section picks up on the unfolding formula 21 relating the Adjoint operator Θ_f to integrating over the horocycle associated to the cusp Γ_∞ of Γ. If f is the characteristic function of a ball, it is supported on $A'K$ with $A' \subset A$ compact and increasing. This leads to an average over the pushed horocycle of a test function, which in turn equidistribute, see Lemma 3.4.

3.1. **The average counting problem.** Let $H = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(v)$, then $H = gUg^{-1}$ with $U = \{\left[\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right]\}$ and some $g \in G$. Identify the orbit $H\Gamma$ in G/Γ with $Y = H/(H\cap\Gamma)$ and normalize the natural measure m_Y to be a probability measure. Then Y defines a closed horocycle in G/Γ and and by a result of Dani [?] it is known that $g_* m_Y$ equidistribute to m_X as $gH \to \infty$ in G/H. In Theorem 3.3 we will cite an effective version of this statement.

The homogeneous space G/H is identified with $\mathbb{R}^2 \setminus \{0\}$ and upon normalizing $m_{G/H}$ to be the restriction of the usual Lebesgue measure on $\mathbb{R}^2$, we force a normalization to $m_{G/(\Gamma\cap H)}$ for the Fubini formula $m_{G/(\Gamma\cap H)} = m_{G/H} m_Y$ to hold. We normalize $m_{G/\Gamma}$ to be the push forward measure of $m_{G/(\Gamma\cap H)}$. We also put $b_t = ga_tg^{-1} \in B = gAg^{-1}$ and $\nu_t = b_{t*}m_Y$ where $a_t = \mathrm{diag}\left(t, t^{-1}\right)$ and $A = \{\left[\begin{smallmatrix}s & 0\\ 0 & s^{-1}\end{smallmatrix}\right] : s \in \mathbb{R}^\times\}$.

Introduce

$$F_{t,I}(g\Gamma) = \frac{|g\Gamma v \cap B_{t,I}|}{m_{G/H}(B_{t,I})}$$

so that $F_{t,I}(\Gamma) = \frac{1}{|I|T^2}N_{T,I}$.

If $K = \mathrm{SO}_2(\mathbb{R})$ we have the Iwasawa decomposition $G = KAU = KBH$. Using this decomposition we define $B_t \subset \mathbb{R}^2 \setminus \{0\}$:

$$B_t = K\{a \in A : \|a\| \le t\}e_1 = K\{b \in B : \|b\| \le \tilde{t}\}v$$

where $\|\cdot\|$ denotes the maximum norm and $\tilde{t}$ is seen to be linear in t. Then B_t agrees with the euclidean annulus $\{t^{-1} \le x \le t : x \in \mathbb{R}^2\}$. To account for counting in sectors, we define k_θ such that $k_\theta e_1 = (\sin\theta, \cos\theta)^T$ and for any interval of angles $I = [\theta_1, \theta_2]$ we let $K(I) = \{k_\theta \in K : \theta_1 \le \theta \le \theta_2\}$. It will be useful to rewrite $K(I)$ as $k_{\theta_0}K([-\theta,\theta])$ for some θ_0, θ. We define the sector of an annulus by

$$B_{t,I} = K(I)\{a \in A : \|a\| \le t\}e_1.$$

Since by assumption Γv is discrete, there exists t_0 such that for all $t > t_0$,

$$\{w \in \Gamma v : \|v\| \le t, \measuredangle(w, e_2) \in I\} = \Gamma v \cap B_{t,I}.$$

We proceed by upgrading the last estimate to a pointwise bound. For each t, we pick ϕ_ε (where $\varepsilon = \varepsilon(t)$ will be chosen later) to be a smooth function of support contained in $\{g \in G : \|g - e\|, \|g^{-1} - e\| \le \varepsilon\}\Gamma$ satisfying $m_{G/\Gamma}(\phi_\varepsilon) = 1$. To prove the inequalities (17) we need to relate $\int_{G/\Gamma} F_{t,I}\phi_\varepsilon dm_{G/\Gamma}$ with $F_{t,I}(\Gamma)$ For this, we show effective well-roundedness of the sets $B_{t,I}$.

3.2. **Wellroundedness.** By replacing ϕ_ε by $k \cdot \phi_\varepsilon$ for a correctly chosen $k \in K$ such that $F_{t,I}(kg\Gamma) = \frac{|g\Gamma v \cap k^{-1}B_{t,I}|}{m_{G/H}(B_{t,I})}$ counts points of $g\Gamma v$ contained in a sector that is symmetric around the y-axis. Furthermore, we may divide I (into at most 4 subsectors) to assume that $I \subset [-\pi/4, \pi/4]$. The multiplication map $K \times A \times U \to G$ is locally bilipschitz, so that for $g \in \mathrm{Supp}(\phi_\varepsilon)$, we have $g = k(g)a(g)u(g)$ with $\|k(g) - e\|, \|a(g) - e\|, \|u(g) - e\| \ll \varepsilon$. In fact, we chose bilipschitz coordinates $[-\varepsilon, \varepsilon] \to K_\varepsilon \subset K$, $x \mapsto \begin{bmatrix} \sqrt{1-x^2} & x \\ -x & \sqrt{1-x^2} \end{bmatrix}$, $[1-\varepsilon, 1+\varepsilon] \to A_\varepsilon \subset A$, $x \mapsto \left[\begin{smallmatrix} x & 0 \\ 0 & 1/x \end{smallmatrix}\right]$ and $[-\varepsilon, \varepsilon] \to U_\varepsilon \subset U$, $x \mapsto \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$ so that $\mathrm{Supp}(\phi_\varepsilon) \subset K_\varepsilon A_\varepsilon U_\varepsilon$.

Lemma 3.1. *If $k \in K_\varepsilon$ then $kB_{s,[-\theta,\theta]} \subset B_{s,[-\theta-\varepsilon,\theta+\varepsilon]}$, and there exists a constant $c > 0$ so that for all $\varepsilon \ll 1$, we have that if $a \in A_\varepsilon$ then $aB_{s,[-\theta,\theta]} \subset B_{s+s\varepsilon,[-\theta-c\varepsilon,\theta+c\varepsilon]}$, and if $u \in U_\varepsilon$ then $uB_{s,[-\theta,\theta]} \subset B_{s+s\varepsilon,[-\theta-c\varepsilon,\theta+c\varepsilon]}$.*

Proof. The claim for $k \in K_\varepsilon$ follow from the fact that K acts by rotation.

The symmetry for $a \in A_\varepsilon$ is such that a either stretches along the vertical direction, in which case the angle is squeezed or it contracts in the vertical direction and the angle is increased. For any $w \in B_s$, $\|aw\| \le \|w\|\|ae_2\| \le (1+\varepsilon)\|w\|$ so that the latter case remains. Let w be the boundary vector of $B_{t,I}$ in the first quadrant, thus $w = (t\sin\theta, t\cos\theta)$. If θ' denotes the angle of aw, then $\tan\theta' = \frac{1+\varepsilon}{1-\varepsilon}\tan\theta \le (1+2\varepsilon)\tan\theta \le \tan(1+2\varepsilon)\theta \le \tan(\theta + 2\varepsilon)$. By symmetry around the y-axis, this proves the second case.

Finally, we turn to the action of $u \in U_\varepsilon$. If $u = \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$, and $w = (s\sin\vartheta, s\cos\vartheta)$ for some $\vartheta \le \theta$ (and by symmetry we may assume that $x, \vartheta \ge 0$), then $\|uw\|^2 \le s^2(1 + 2x\sin\vartheta\cos\vartheta + x^2\cos^2\vartheta)) \le s^2(1+3x)$ and so $\|uw\| \le (1+2\varepsilon)\|w\|$. For the new angle ϑ' of uw, we have that $\tan\vartheta' = \tan\vartheta + x \le \tan(\vartheta + x) \le \tan(\theta + \varepsilon)$.

□

Lemma 3.2. *There exists a constant $c_2 > 0$ such that for $\varepsilon \ll 1$, for any $g \in \mathrm{Supp}(\phi_\varepsilon)$,*

$$gB_{t-c_2\varepsilon t,[-\theta+c_2\varepsilon,\theta-c_2\varepsilon]} \subset B_{t,I} \subset gB_{t+c_2\varepsilon t,[-\theta-c_2\varepsilon,\theta+c_2\varepsilon]}$$

Proof. We iterate the previous lemma with respect to the decomposition $g = k(g)a(g)u(g)$ to prove the second inclusion. The first inclusion follows from the second upon multiplying with g^{-1} and appropriate substitions for t and θ. □

We give notion of wellroundedness of sectors, which takes the following form. For any g an ε-ball of $\mathrm{SL}_2(\mathbb{R})$

$$(1+\mathcal{O}(\varepsilon))\langle \tilde{N}_{T-c_2T\varepsilon,I-c_2\varepsilon}, \phi_\varepsilon\rangle \le \frac{1}{|I|T^2}N_{T,I}(g) \le (1+\mathcal{O}(\varepsilon))\langle \tilde{N}_{T+c_2T\varepsilon,I+c_2\varepsilon}, \phi_\varepsilon\rangle \tag{17}$$

where $\tilde{N}_{T,I}(g) = \frac{|g\Gamma v \cap B_{T,I}|}{|B_{T,I}|}$ denotes the normalized counting function and $\phi_\varepsilon \in C_c^\infty(G/\Gamma)$ is supported on an ε-neighbourhood of $\Gamma \in G/\Gamma$ with $m_{G/\Gamma}(\phi_\varepsilon)$. See Section 3.1 for a proof.

We proceed with the proof inequalities (17).

Proof of inequalities (17). For the (centered) interval I, we denote $I + c_2\varepsilon$ the thickened interval $[-\theta - c_2\varepsilon, \theta + c_2\varepsilon]$ and by $I - c_2\varepsilon$ the shrunken interval $[-\theta + c_2\varepsilon, \theta - c_2\varepsilon]$. By Lemma 3.2, and the fact that $m_{G/\Gamma}(\phi_\varepsilon) = 1$,

$$\frac{m_{G/H}(B_{t-t\varepsilon c_2, I-c_2\varepsilon})}{m_{G/H}(B_{t,I})} \int F_{t-c_2t\varepsilon, I-c_2\varepsilon}\phi_\varepsilon dm_{G/\Gamma} \le F_{t,I}(\Gamma)$$
$$\le \frac{m_{G/H}(B_{t+c_2t\varepsilon, I+c_2\varepsilon})}{m_{G/H}(B_{t,I})} \int F_{t+c_2t\varepsilon, I+c_2\varepsilon}\phi_\varepsilon dm_{G/\Gamma}$$

We conclude

$$F_{t,I}(\Gamma) = \frac{|I + c_2\varepsilon|(t + c_2t\varepsilon)^2}{|I|t^2} \int F_{t+c_2t\varepsilon, I+c_2\varepsilon}\phi_\varepsilon dm_{G/\Gamma} = (1 + \mathcal{O}(\varepsilon)) \int F_{t+c_2t\varepsilon, I+c_2\varepsilon}\phi_\varepsilon dm_{G/\Gamma}.$$

and a similar statement from below. □

3.3. **Equidistribution of Horocycles implies Counting.** It remaines therefore to show equidistribution of F_t as t gets large.

For the next theorem, we define a Sobolev norm $\mathcal{S}_d$. Fix a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{sl}_2(\mathbb{R})$ and put

$$\mathcal{S}_d(f) = \sum_{\deg D \le d} \|Df\|_{L^2(m_{G/\Gamma})}.$$

The following Theorem is phrased in terms of the height y of the horocycle when mapped to the cusp at infinity. We recall that the measure $\nu_t = b_{t*}m_Y$ is supported on a closed horocycle of length $t = 1/y^{\frac{1}{2}}$.

Theorem 3.3 ([Sar][Theorem 1], for K-invariant functions also [Strom04][Theorem 3..?])**.** *There exists constants $c = c_v > 0$ and $\kappa_1 = \kappa_1(\Gamma) > 0$ such that for any $f \in C_c^\infty(G/\Gamma)$*

$$|\nu_t(f) - \frac{1}{m_{G/\Gamma}(G/\Gamma)} m_{G/\Gamma}(f)| \le c(t^{-1}\mathcal{S}_4(f) + t^{-(2-\kappa_1)}\mathcal{S}_0(f))$$

The term $t^{-(2-\kappa_1)}\mathcal{S}_0(f)$ is absent by lack of cuspidal eigenvalues. By [Strom04]*, the bound $c(t^{-1}\mathcal{S}_4(f) + t^{-(2-\kappa_1)}\mathcal{S}_0(f))$ can be replaced by $c_\delta(t^{-1+\delta}\mathcal{S}_2(f) + t^{-(2-\kappa_1)+\delta/2}\mathcal{S}_0(f))$ for any $\delta > 0$.*

Lemma 3.4. *For any $\alpha \in C_c^\infty(G/\Gamma)$*

$$\int_{G/\Gamma} F_{t,I}\alpha dm_{G/\Gamma} = \frac{1}{m_{G/\Gamma}(G/\Gamma)} m_{G/\Gamma}(\alpha) + \mathcal{O}\left(t^{-1+\delta}\mathcal{S}_2(\alpha) + t^{-(2-\kappa_1)+\delta}\mathcal{S}_0(\alpha)\right)$$

with explicit and implicit constants as in the previous Theorem.

Proof. By the unfolding argument for $m_{G/(\Gamma\cap H)} = m_{G/H}m_Y$ (see [**?**][Proof of Theorem 5.1]), we have that for any $\alpha \in C_c^\infty(G/\Gamma)$

$$\int_{G/\Gamma} F_{t,I}\alpha dm_{G/\Gamma} = \frac{1}{m_{G/H}(B_{t,I})} \int_{B_{t,I}} g_*m_Y(\alpha) dm_{G/H}(gH).$$

Recall that $m_{G/H}$ is normalized to be compatible with the usual Lebesgue measure on $\mathbb{R}^2$ and thus, in radial coordinates, $gH = w = kb_sv$ for some k and $s = \|w\|/\|v\|$ so that $g_*m_Y = k_*\nu_s$.

$$m_{G/H}(B_{t,I}) \int_{G/\Gamma} F_{t,I}\alpha dm_{G/\Gamma} =$$
$$\int_{K(I)} \int_0^\infty \left(\frac{1}{m_{G/\Gamma}(G/\Gamma)} m_{G/\Gamma}(k \cdot \alpha) + \mathcal{O}\left(r^{-1+\delta}\mathcal{S}_2(k \cdot \alpha) + r^{-2+\kappa_1+\delta/2}\mathcal{S}_0(k \cdot \alpha)\right)\right) dkrdr.$$

As $\mathcal{S}(k \cdot \alpha) \ll \mathcal{S}(\alpha)$,

$$\int_{G/\Gamma} F_{t,I}\alpha dm_{G/\Gamma} = \frac{1}{m_{G/\Gamma}(G/\Gamma)} m_{G/\Gamma}(\alpha) + \mathcal{O}\left(t^{-1+\delta}\mathcal{S}_2(\alpha) + t^{-2+\kappa_1+\delta/2}\mathcal{S}_0(\alpha)\right)$$

□

3.4. **General Shapes.** We wish to quantify a version of this theorem to obtain polynomial error bounds for more general shapes than sectors and balls.

Theorem 3.5. *For any ϕ_Ω be the characteristic function of a bounded set $\Omega \subset \mathbb{R}^2$ whose boundary is piecewise smooth.*

$$\Theta_{T\phi_\Omega} = c(\Gamma, v)|\Omega|T^2 + \mathcal{O}_\Omega(T^{2-\kappa})$$

Lemma 3.6. *If suffices to proof Theorem 3.5 for conal sets $\Omega = \{t\rho(\theta) : t \in (0,1], \theta \in S^1\}$ where $\rho : S^1 \to (0,1]$ is a piecewise smooth function.*

Proof. Consider a closed set $\Omega \subset \mathbb{R}^2$ whose boundary is piecewise smooth and may assume that Ω is contained in the unit ball. We may restrict to a connected component of Ω and may assume it to be simply connected, since else we first count in each bounded connected component of Ω^c. In fact, we shall do a finer geometrically reduction as such: We can do a decomposition into finitely many sectorial shells of Ω to assume that there are two funtions ρ_1, ρ_2 globally defined in polar coordinates giving top and bottom boundary - either closing up or connected with a radial part of a sector. Thus if we can count under each curve $T\rho_1$, $T\rho_2$, we can collect a count of Ω by inclusion exclusion. It is important to realize that this decomposition is independent of T, that is, each component has the same asymptotics growth in T. □

We clearly have to extend Lemma 3.2.

Lemma 3.7. *It suffices in Lemma 3.6 to assume that ρ is smooth. More precisely, there exists smooth $\rho_\delta^\pm$ such that for $\Omega_\delta^\pm = \{t\rho_\delta^\pm(\theta) : t \in (0,1], \theta \in S^1\}$*

$$\Omega_\delta^- \subset \Omega \subset \Omega_\delta^+$$

and $m_{\mathbb{R}^2}(\Omega_\delta^+ \setminus \Omega_\delta^-) = \mathcal{O}_\rho(\delta)$, $\mathcal{S}_*(\rho_\delta^\pm) = \mathcal{O}_\rho(\delta^{-1/2})$.

Proof. This is the first smoothing argument. Each discontinuity can be replaced by changing ρ on a δ-neighborhood, with $\|\rho'\|_\infty = \mathcal{O}(\delta^{-1})$ on this neighborhood. □

We just write $\rho_\delta = \rho_\delta^\pm$.

Lemma 3.8. *There exists ρ_δ^ε with $\Omega_\delta^\varepsilon = \{t\rho_\delta^\varepsilon(\theta) : t \in (0,1], \theta \in S^1\}$ such that for any $g \in B_\varepsilon^G$,*

$$gT\Omega \subset T\Omega_\delta^\varepsilon$$

and $m_{\mathbb{R}^2}(\Omega_\delta^\varepsilon \setminus \Omega_\delta) = \mathcal{O}_\rho(\varepsilon\delta^{-1})$.

Proof. Consider the Cartan decomposition $G = KAK$ and the associated local diffeomorphism (in particular bilipschitz) $S' \times \mathbb{R}_{\geq 0} \times S' \to G$ given by $k(\theta_1)a(s)k(\theta_2) = g$ (notation as before Lemma 3.1) where $\theta_1, \theta_2, s-1 = \mathcal{O}(\varepsilon)$. In G/N-coordinates $\phi_{T\Omega} = g(\phi_T(\theta, t))$ becomes $k(\theta)a(\rho(\theta))a(t)N$, $t \leq T, \theta \in S^1$. Applying the ε-deformation

$$g\phi_{T\Omega}(\theta, t) = k(\theta_1)a(s)k(\theta + \theta_2)a(\rho(\theta))a(t)N$$

To find the parameter θ', t' for which $k(\theta')a(\rho(\theta)t')N = a(s)k(\theta + \theta_2)a(\rho(\theta))a(t)N$ we have to estimate $a(s)ka(s)^{-1} = kan$ in Iwasawa coordinates. Bilipschitzness of the local coordinate system, and polynomiality of the Adjoint action on $\mathfrak{sl}_2(\mathbb{R})$ we get

$$a(s)ka(s)^{-1} = kk_\varepsilon a_\varepsilon n_\varepsilon$$

for $k_\varepsilon, a_\varepsilon, n_\varepsilon \in B_\varepsilon^G$. N is normal in AN so that

$$g\phi_{T\Omega}(\theta, t) = k(\theta_1 + \theta + \theta_2 + \mathcal{O}(\varepsilon))a(\rho(\theta)t(s + \mathcal{O}(\varepsilon))N = k(\theta)k_\varepsilon a_\varepsilon a(\rho(\theta))a(t)N$$

for some (new) $k_\varepsilon, a_\varepsilon \in B_\varepsilon^G$ and define θ_ε such that $k(\theta_\varepsilon) = k(\theta)k_\varepsilon$, and t_ε such that $a(t_\varepsilon) = a(\rho(\theta_\varepsilon))^{-1}a(\rho(\theta))a_\varepsilon$

$$g\phi_{T\Omega}(\theta, t) = k(\theta_\varepsilon)a(\rho(\theta_\varepsilon))a(t)a(t_\varepsilon)N$$

We note that $t_\varepsilon = t_\varepsilon(\theta)$ clearly depends on θ but also on ρ^δ, more precisely

$$t_\varepsilon = 1 + \mathcal{O}_\rho(\varepsilon)$$

if θ is $\max(\varepsilon, \delta)$-away from the discontinuities of ρ, and

$$t_\varepsilon = 1 + \mathcal{O}_\rho(\delta^{-1}\varepsilon)$$

on $\max(\varepsilon, \delta)$-neighborhoods around the discontinuities of ρ. Thus take $\rho_\delta^\varepsilon(\theta) = \rho_\delta(\theta)t_\varepsilon(\theta)$. □

The other inequality for wellroundedness can either be obtained by redoing the steps in the proof, or better, by applying the Lemma for ρ replaced by ρ_δ^-.

Theorem 3.5 now follows from equation 3, see Lemma 3.4 for details.

4. A combination of Section 2 and 3: Bounding directly with the Maass-Selberg-Relations

Since

$$\Delta_m E_m(g,s) = s(1-s)E_m(g,s),$$

for the Laplace operator Δ_m given in Iwasawa coordinates by $\Delta_m = -y^2\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right) + imy\frac{d}{dx}$ we find by integration by parts,

$$\langle E_m(s), \psi_\varepsilon\rangle = \frac{1}{s(1-s)}\langle \Delta_m E_m(s), \psi_\varepsilon\rangle = \frac{1}{s(1-s)}\langle E_m(s), \Delta_m\psi_\varepsilon\rangle. \tag{18}$$

Thus, repeated application of Δ will show that the expression $\langle E_m(s), \psi_\varepsilon\rangle$ has super polynomial decay in the variable s, which comes as a replacement for Lemma 9.2. In fact, we can once more argue by convexity to upgrade for non-integer values. See Lemma 4.2 below.

As described in the introduction, one has the following spectral decomposition of the fourier coefficients of the counting function,

$$N_{T,m}(g) = \frac{1}{2\pi}\int_{(c)} \frac{E_m(g,s)}{s} R^{2s} ds \tag{19}$$

which we shall integrate against ψ_ε. Since the support of ψ_ε is compact, we may use truncated Eisenstein series $\tilde{E}_n^Y$ when necessary. We note that Sarnak obtains

Lemma 4.1 (Sarnak, Horocycles,(Lemma 2.10 and formulas (2.9)).

$$\|\tilde{E}_n^Y\|_2 \ll \log n \|\tilde{E}^Y\|_2$$

so that Theorem 13.2 also bounds the weighted Eisenstein series.[5] Indeed, this may be done by considering the analogous Maass-Selberg relations. Combining with formula 18

$$|\langle E_m(s), \psi_\varepsilon\rangle|\,|s|^{-1} \ll \frac{1}{|s|^3}\varepsilon^{-d}\log m\alpha(t)^{1/2} \tag{20}$$

where ε^{-d} essentially bounds the Sobolev norm $\|\Delta\psi_\varepsilon\|_2$ and $\int_{-T}^{T}\alpha(t)dt = \mathcal{O}(T^2)$ by Theorem 13.2.[6]

Interchanging integrals in the expression $\langle E_m(s), \psi_\varepsilon\rangle$ where we apply formula 19, we shift contour [7] to $s = \frac{1}{2} + it$, pick up the possible residue $\delta_{m0}\operatorname{res}_{s=1}\langle E_m(s), \psi_\varepsilon\rangle$ (the main term) and the error term

$$\frac{1}{2\pi}\int_{(1/2)} \frac{|\langle E_m(s), \psi_\varepsilon\rangle|}{s} R^{2s} ds = R\varepsilon^{-d}\log m \int_{-\infty}^{\infty}\frac{\alpha(t)^{1/2}}{1+|t|^3}dt.$$

The the last integral is absolute convergent by the same argument as in the proof of Lemma 2.8.

Note that it would suffice to have $\frac{1}{|s|^3}$ replaced by $\frac{1}{|s|^{3/2}}$ for which one again argues by convexity, and resulting in better error terms.

Lemma 4.2. *Bound on*

$$|\langle E_m(s), \psi_\varepsilon\rangle| \leq \frac{1}{|s|^k}$$

for $k \in (1,3)$ with better ε-dependence.

Proof. Like the proof of Lemma 9.2 □

5. Selberg's work on Eisenstein Series

Selberg [Indian, Goettingen] proved the following theorem about the Eisenstein series $E(z,s)$.

Theorem 5.1. *Let Γ have a single cusp at ∞.*

A: *The function $E(z,s)$ is absolutely convergent for* $\operatorname{Re} s > 1$ *and is is uniformly bounded on* $\operatorname{Re} s \geq 2$.

M: *There is a meromorphic extension to all of* $\mathbb{C}$.

1/2: *E is holomorphic on* $\operatorname{Re} s = 1/2$.

E: *$E(z,s) = \phi(s)E(z,1-s)$ for a meromorphic function $\phi(s)$.*

ϕ: *$\phi(s)$ is holomorpohic and unitary on* $\operatorname{Re}(s) = 1/2$.

ω: *$\omega(t) = 1 - \frac{\phi'}{\phi}(1/2+it)$ is even, ≥ 1 and $\int_T^T \omega(t)dt \ll_\Gamma T^2$.*

[5] This bound is much better than the one from MS, I think one can probably also improve on Theorem 2.7.

[6] Maybe Sarnak forget's a squareroot saving here

[7] again by the polynomial bound on average argument, but here bounds are provided for all of $\operatorname{Re} s \geq 1/2$

P: *All poles of E in* $\operatorname{Re} s > 1/2$ *lie in* $(1/2, 1]$ *and all are simple.*
1: *There is always a pole at* $s = 1$ *with residue equal to* c_Γ.
G1: $E(z,s) \ll_\Gamma \mathcal{O}(\omega(t)^{1/2} e^{3|t|-2\pi y})$ *for s away from poles and* $\operatorname{Re} s \in [\frac{1}{2}, 3/2]$
G1/2: $\int_T^T |E(z, 1/2+it)|^2 dt \ll_\Gamma T^2 + Ty$.

For the first facts see Theorem 7.3 [Selberg, Goettingen, p653] and Theorem 11.8 [Hej2, p130]. [G1] is (8.17) in [Selberg, Goettingen, p658] and Theorem 12.9 (d) [Hej2, p164]. [1/2] is (3) in [Selberg, Goettingen, p627] and Prop 7.2 in [Iwaniec, p101].

As remarked before, for $c > 1$,

$$\Theta_\psi(z) = \frac{1}{2\pi i} \int_{(c)} \Psi(-s) E(z,s) ds.$$

Assuming E is "sensible" at ∞ we can shift contours to $\operatorname{Re} s = 1/2$, see discussion in [Hej2, p82 top]. Indeed, Lemma 6.1 shows that the average behaviour given by $[G1], G[1/2]$ suffices.

6. The Contour Shift argument

We need a slightly different version of Proposition 9.3 and Lemma 9.4.

Lemma 6.1. *Let ϕ be smooth and compactly supported. Let $F(s)$ be meromorphic on $\mathbb{C}$ and is holomorphic on the right hand side of (a) including (a) itself, except for finitely many poles $\{s_j\}$, say in $R[a,b] = \{s :\in \mathbb{C} : \operatorname{Re}(s) \in (a,b)\}$. Assume further that F is bounded on $(b) \cup R[b,\infty]$ and uniformly in the real parameter, satisfies $\int_{-T}^T |F(a+it)|^2 dt = \mathcal{O}(T^2)$, and $F(s) = \mathcal{O}(\omega(t)e^{3t})$ for $s = \sigma + it \in R[a,b]$ with $\int_{-T}^T \omega(t)^2 = \mathcal{O}(T^2)$ for a positive even function $\omega(t) \geq 1$. Then*

$$\frac{1}{2\pi i} \int_{(b)} F(s)\Psi(s) ds = \sum_{s_j} \operatorname{Res}_{s_j}(F)\Psi(s_j) + \frac{1}{2\pi i}\int_{(a)} F(s)\Psi(s) ds$$

Proof. We only have to modify the two preceding proofs. The left hand side exists because $\Psi(s)$ is rapidly convergent and $F(s)$ is bounded. By partial integration and Cauchy-Schwarz (as in the argument below equation 14), we also see that $\frac{1}{2\pi i}\int_{(b)} F(s)\Psi(s)ds$ is absolutely convergent. Let C be the maximum of the implicit constants of the growth assumption on F, ω

$$\int_{-T}^T |F(a+it)|^2 dt \leq CT^2, \int_{-T}^T \omega(t)^2 \leq CT^2, |F(s)| \leq C\omega(t)e^{3t}, |F(s)| \leq C \ (\operatorname{Re}(s) \geq b).$$

Introduce the auxilary function

$$H(s) = \int_0^1 d\tau \int_a^b d\xi F(s+\xi+i\tau), \quad s = \sigma + it$$

which is holomorphic for $|t| \geq T_0$ (only depending on the location of poles of F), or more generally

$$H_\rho(s) = \int_{[a,b]+i[0,1]} F(s+w)\rho(w) dw$$

for $\rho : [a,b] + i[0,1] \to B_1^{\mathbb{C}}(0)$. Holomorphicity follows e.g. by interchange of integral and $\partial_{\overline{z}}$. We shall obtain a uniform among such, ρ, so that one can consider or fixed s, $\rho_{(w)}$ the sign function of $F(s+w)$. Therefore, it suffices to prove $|H_\rho(s)| \ll_C |s|$ to conclude

$$\int_{[a,b]+i[0,1]} |F(s+w)| dw \ll |s|$$

and therefore

$$\int_{[a,b]} |F(s_n + x + yi)| dx \ll |s_n|, \quad w = x + iy$$

for infinitely many $s_n \to \infty$. We will write $H = H_\rho$. We have

$$|H(s)| \leq \int_{[a,b]+i[0,1]} |F(s+z)\rho| dz\tau \ll_{b-a} \left(\int_{[a,b]+i[0,1]} |F(s+z)|^2 dz\right)^{\frac{1}{2}} = \left(_{[a,b]+i[t,t+1]}|F(\sigma+z)|^2 dz\right)^{\frac{1}{2}}$$

The growth rate of $H(s)$ falls into three cases to the regions (a), (b) and $R[a,b]$, (implicit constants only depending on C)

$$|H(s)| \ll t \quad s \in (a) \cup (b)$$
$$|H(s)| \ll te^{3t} \quad s \in R[a,b].$$

Introduce $\varepsilon > 0$. There exists $\theta_0 \in [0, 2\pi]$ and $T_\varepsilon \geq T_0$, such that for any

$$s = \sigma + it = re^{i\theta_s} \in R[a,b] \cap \{\mathrm{Im}(s) \geq T_\varepsilon\},$$

we have $|\theta_s - \theta_0| \leq \varepsilon$. Using this, we may also find $m \in \mathbb{N}_{>1}$ such that

$$\cos m\theta_s < -1 + \varepsilon$$

(after possibly increasing T_ε). ε will be fixed from now on. Introduce $\delta > 0$. There exists $T_\delta > T_\varepsilon$ such that for any $s \in R[a,b] \cap \{\mathrm{Im}(s) \geq T_\delta\}$,

$$|H(s)e^{\delta s^m}| = |H(s)e^{\delta r^m \cos m\theta_s}| \ll te^{3t + \delta r^m \cos m\theta_s} \leq 1$$

since $r > t$, and $\cos m\theta_s$ uniformly close to -1. Dividing by s, we see that $H(s)s^{-1}e^{\delta s^m}$ is now uniformly bounded on the boundary of $R[a,b] \cap \{T_\varepsilon \leq \mathrm{Im}(s) \leq T_\delta\}$, and by the maximum modulus principle also in its interior, and thus

$$|H(s)| \ll |se^{\delta s^m}|$$

for any $s \in ((a) \cup (b) \cup R[a,b]) \cap \{T_\varepsilon \leq \mathrm{Im}(s) \leq T_\delta\}$. Letting $T_\delta \to \infty$, $|H_T(s)| \ll |se^{\delta s^m}|$ for any $T_\varepsilon \leq \mathrm{Im}(s)$ in the strip. Since δ was arbitrary, we may take $\delta \to 0$ to see that

$$|H(s)| \ll t$$

for any $t > T_\varepsilon$. This argument may be repeated for the negative strip.

Thus we have shown that there exists a sequence $\gamma_n^\pm$ lines, connecting (a) with (b) at above $\mathrm{Im}\, s = t_n$ (resp. $\mathrm{Im}\, s < -t_n$) for $t_n \to \infty$ for which $\int_{\gamma_n} |F(s)| \ll t_n$.

We can deduce absolute convergence of $\frac{1}{2\pi i}\int_{(c)'} F(s)\Psi(s)ds$ for $(c)'$ stands for the line c possible with a neighbourhood around the pole removed.

We also can justify the countour shift: Exploiting super-polynomial decay of Ψ, we can use Cauchy's residue theorem to show that the integral of $\Psi(s)F(s)$ along the domain enclosed by $\gamma_n^\pm$ and $(a) \cup (b)$ equals $2\pi i \sum_{s_j} \mathrm{Res}_{s_j}(F)\Psi(s_j)$ with horizontal contribution $\int_{\gamma_n} \Psi(s)F(s)ds$ vanishing as $n \to \infty$. We note that we already know absolute convergence of the integrals $\frac{1}{2\pi i}\int_{(a)-(b)} F(s)\Psi(s)ds$, so that it suffices to take the limit under some subsequence. □

6.1. **Weighted Eisenstein series.** A similar theorem holds for $E_n(z,s)$, in particular $[G1/2]$ is proven in Marklof-Strombergssen [Kronecker] with explicit dependency in n. We are lacking of $[G1]$ however.

7. Fourier decomposition

In view of the spectral theorem, Theorem 2.3, a post-contour shift is already provided to us. However, it remains to identify the Fourier coefficient explicitely. By [Hej2, p291, p317, p773 Note 5]

$$f(z) = \sum d_n \varphi_n(z) + \int_0^\infty g(t)E(z, \frac{1}{2} + it)dt$$

for $f \in C^2 \cap L^2, \Delta f \in L^2$ uniformly and absolutely on compacta where

$$g(t) = \frac{1}{2\pi}\int_{\mathbb{H}/\Gamma} f(z)E(z, \frac{1}{2} + it)d\mu(z).$$

Thus we wish to know that for $f = \Theta_\psi$, $\psi : G/N \to \mathbb{R}$ compactly supported, $g(t) = \Psi(\frac{1}{2} + it)$. We note that unfolding,

$$\int_{\mathbb{H}/\Gamma} \Theta_\psi(z)E(z, \frac{1}{2} + it)d\mu(z) = \int_{\mathbb{H}/\Gamma_\infty} \psi(z)E(z, \frac{1}{2} + it)d\mu(z)$$

$$\int_A \psi(a \cdot i)\int_{N/(\Gamma_\infty \cap N)} E(an \cdot i, \frac{1}{2} + it)dnda$$

Now reference to the Fourier decomposition of E has to be made, Thm 11.6 (F) [Hej, p297], see also line (2.53) [Strom]:

$$E(z, \frac{1}{2} + it) = y^{\frac{1}{2}+it} + \phi(\frac{1}{2} + it)y^{\frac{1}{2}-it} + \sum_{n \neq 0} c_n(t)y^{\frac{1}{2}}K_{it}(2\pi|n|y)e(nx)$$

so that

$$g(t) = \int_A \psi(a(y) \cdot i)\left(y^{\frac{1}{2}+it} + \phi(\frac{1}{2} + it)y^{\frac{1}{2}-it}\right)\frac{dy}{y}$$

which is a sum of $\Psi(s)$ and $\phi(s)\Psi(s)$. Since ϕ is unitary (Theorem 5.1,[ϕ]) (alternatively, use Theorem 5.1,[E]), we can bound both terms with Theorem 5.1,[G1/2].

Derivations of the spectral decomposition following this argument in the opposite direction are sketched in [Ku, p86], [I, p102].

7.1. **Weighted Eisenstein series.** Since we know properties [E],[1/2] for E_k by works of Marklof-Strombergssen, the argument of Section 2.2 that uses equation 7.1.1 is therefore justified.

7.1.1. *Counting in smooth Star shaped domains.* Thus let $\phi_\Omega : \mathbb{R}^2 \to \mathbb{R}$ the characteristic function of a star shaped domain Ω whose smooth boundary is given in polar coordinates $(\rho(\theta), \theta)$. Let $\psi_U^\pm(t) \in C_c^\infty(0,\infty)$ be a smooth bumb function as employed in Theorem 2.8, the domain $T\Omega$ is approximated by $\{T\psi_U^\pm(t)\rho(\theta)k(\theta)e_1 : t \in (0,1], \theta \in S^1\}$,and we let $\phi_{T\Omega,U}^\pm$ defined by

$$\phi_{T\Omega,U}^\pm(t,\theta) = \psi_U^\pm(t/T)\rho(\theta)k(\theta)e_1$$

and similar for $\phi_{T\Omega}$. For star shaped domains we have the easy inclusion

$$\Theta_{\phi_{T\Omega,U}^-}(\Gamma) \leq \Theta_{\phi_{T\Omega}}(\Gamma) \leq \Theta_{\phi_{T\Omega,U}^+}(\Gamma)$$

and it suffices to estimate $\Theta_{\phi_{T\Omega,U}^\pm}(\Gamma)$. We drop the superscript $\pm$.

We have the following Fourier decomposition, where $\rho(m)$ are the Fourier decomposition of ρ along K,

$$\Theta_{\phi_{T\Omega,U}}(g\Gamma) = cm_{\mathbb{R}^2}(\phi_{T\Omega,U}) + \sum_m \left(\int_{\mathrm{Re}(s)=\frac{1}{2}} \widehat{\psi_\varepsilon}(s) E_m(s,g) ds \right) \rho(m) e(m\theta).$$

The extension of equation 15 reads

$$\left| \int_{\mathrm{Re}(s)=\frac{1}{2}} \widehat{\psi_U}(s) E_m(s,g) ds \right| = \mathcal{O}(T^{1/2} U^{1/2+} m)$$

so that we wish to bound $\sum \rho(m)m$, which we immediately see to be $\mathcal{O}(\mathcal{S}_2(\rho))$ and

$$\Theta_{\phi_{T\Omega,U}}(g\Gamma) = cm_{\mathbb{R}^2}(\phi_{T\Omega,U}) + \mathcal{O}(T^{1/2} U^{1/2+}).$$

8. Outlook: Eisenstein-Veech series

Let x be a translation surface and $V(x)$ the set of holonomy vectors of saddle connections on x. Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be of compact support (and may considered as radial function on $\mathbb{R}$). Its Siegel-Veech transform is $\widehat{\psi}(x) = \sum_{v \in V(x)} \psi(\|v\|)$). We define the Eisenstein-Veech transform by $E(x,s) = \sum_{v \in V(x)} \|v\|^{-s}$. Using Veech's proof using Lebesque-Stieltjes integration, and Masur's upper bound this object is well defined for $\mathrm{Re}\, s > 1$ (see discussion around equation 7).

Proposition 8.1 (Veech)**.** *For any x the series $E(x,s)$ is absolutely convergent for* $\mathrm{Re}\, s > 1$.

To each x there is an hyperbolic plane $\mathbb{H}_x$ attached via the $\mathrm{SL}_2(\mathbb{R})$-action and forgetting the north direction of the translation surface. In particular, we can study the Laplacian on $\mathbb{H}_x$. By a theorem of Athreya-Masur-Cheung, the Siegel-Veech transform is in L^2 for compactly supported functions.

Question. Is it possible to define a regularized series $E(x,s)$ which becomes square integrable? Is it possible to then adapt Veech's argument to calculate also its residue at $s = 1$ (conjecturally equal to the Siegel-Veech constant)? Is it possible to use Dozier's uniform of Masur's theorem to deduce uniform convergence of $E(x,s)$? Is it possible to use the result of NRW to meromorphically extend $E(x,s)$ to every $y \in \mathbb{H}_x$ and almost every x?

The an answer to the first question would lead to meromorphic continuation outside possible poles by the method of de Verdiere/ Goldfeld-Sarnak discussed in Section 11.

9. Appendix: Some complex analysis

9.1. **Mellin transform.** [Following Bergeron: The spectrum of hyperbolic surfaces]

Proposition 9.1. *Let $\phi : \mathbb{R} \to \mathbb{R}_+$ continuous, its Mellin transform is defined by*

$$\Psi(s) = \int_0^\infty \psi(y) y^{s-1} dy, \quad s \in \mathbb{C}.$$

Suppose $\Psi(s)$ is absolutely convergent for some s_0. Then it is absolutely convergent on some half strip $R[a,b] =_{\mathrm{def}} \{s \in \mathbb{C} : a < \mathrm{Re}\, s < b\}$ containing s_0. The inverse formula

$$\phi(y) = \frac{1}{2\pi i} \int_{(c)} \Psi(s) y^{-s} ds.$$

holds for any $\sigma \in R[a,b]$, and the notation (σ) means to integrate along the vertical line $\{c+ir : r \in \mathbb{R}\}$.

Proof. Assume that $\Psi(s_0)$ is absolutely convergent, and consider the following splitting:

$$|\int_0^\infty \psi(y)y^{s-1}dy| \leq \int_0^\infty |\psi(y)|y^{\operatorname{Re} s-1}dy = \int_0^1 |\psi(y)|y^{\operatorname{Re} s-1}dy + \int_1^\infty |\psi(y)|y^{\operatorname{Re} s-1}dy.$$

The first integral is monotonically decreasing in $\operatorname{Re} s$, where as the second one is monotonically increasing, so that they converge on $R[\operatorname{Re} s_0, \infty)$ and $R[0, \operatorname{Re} s_0]$ respectively. Varying s_0 among the the domain of absolutely convergence, we see that this domain is indeed a half-strip. For the inverse formula, let $s = \sigma + ir \in R[a,b]$. We wish to reduce to the Fourier inverse formula. Substitute $y = e^t$, then

$$\Psi(s) = \int_{-\infty}^\infty \psi(e^t)e^{(s-1)t}(e^t dt) = \int_{-\infty}^\infty \psi(e^t)e^{st}dt = \int_{-\infty}^\infty \psi(e^t)e^{\sigma t}e^{irt}dt = \widehat{\psi_\sigma}(r)$$

for

$$\psi_\sigma(t) = \psi(e^t)e^{\sigma t}.$$

Absolute convergence implies $\psi_\sigma \in L^1$ and the Fourier transform is defined. Fourier inversion formula (which applies since $\psi_\sigma \in L^2$ by continuity!) reads now

$$\psi_\sigma(t) = \frac{1}{2\pi i}\int_{-\infty}^\infty \Psi(s)e^{-irt}dt = \frac{1}{2\pi i}\int_{-\infty}^\infty \Psi(s)e^{-irt}dt$$

so that

$$\psi(e^t) = \frac{1}{2\pi i}\int_{-\infty}^\infty \Psi(s)e^{-\sigma t-irt}dt = \frac{1}{2\pi i}\int_{-\infty}^\infty \Psi(s)(e^t)^{-s}dt$$

which is the claimed Mellin transform. □

9.2. **Convexity principles.**

Lemma 9.2.

$$\Psi_U(s) = \begin{cases} s^{-1} + \mathcal{O}(U^{-1}), & \text{as } U \to \infty \\ \mathcal{O}\left(\frac{1}{|s|}\left(\frac{U}{1+|s|}\right)^k\right), & \text{as } |s| \to \infty \end{cases}$$

for any $k > 0$.

Proof. For $k \in \mathbb{N}$, partial integration gives immediately that

$$\Psi_U(s) = \int_0^\infty \psi_U(y)y^{s-1}dy = \mathcal{O}\left(\frac{1}{|s|}\left(\frac{U}{|s|}\right)^k\right).$$

where I replaced $(1+|s|)$ with $|s|$ since we care for this bound as $s \to \infty$. Introducing constants for the $\mathcal{O}(\cdot)$ notation,

$$|f(s)| \leq c_k U^k |s|^{-(k+1)}.$$

A convex combination for k and $k+1$ gives, for any $t \in [0,1]$

$$|f(s)| \leq tc_k U^k|s|^{-(k+1)} + (1-t)c_{k+1}U^{k+1}|s|^{-(k+2)} = U^k|s|^{-(k+1)}\left(tc_k + (1-t)c_{k+1}U/|s|\right)$$

We can therefore extend the bound to arbitrary $n = k + \varepsilon$, if we can find t such that

$$tc_k + (1-t)c_{k+1}U/|s| \leq U^\varepsilon/|s|^\varepsilon.$$

Solving for t,

$$t \leq \frac{U^\varepsilon/|s|^\varepsilon - c_{k+1}U/|s|}{c_k - c_{k+1}U/|s|}$$

Thus, for sufficiently large s, this expression is ensured to be positive (since $c_{k+1}U/|s|$ is very small) , and to be less then one (since $U^\varepsilon/|s|^\varepsilon/c_k$ is very small). Setting $t = \frac{U^\varepsilon/|s|^\varepsilon - c_{k+1}U/|s|}{c_k - c_{k+1}U/|s|}$ provides

$$|f(s)| \leq U^{k+\varepsilon}|s|^{-(k+\varepsilon+1)}.$$

for s sufficiently large (depending on ε and U). □

Proposition 9.3 (Phragmen-Lindeloef-Principle)**.** *Let $f(s)$ be holomorphic on $R[\sigma_1, \sigma_2] \cap \{\mathrm{Im}(s) > c\}$ satisfying the following asymptotics:*

$$f(\sigma + it) = \mathcal{O}(e^{t^\alpha}) \quad \sigma \in [\sigma_1, \sigma_2] \quad \textit{for some } \alpha > 0$$
$$f(\sigma + it) = \mathcal{O}(t^M) \quad \sigma = \sigma_1, \sigma_2 \quad \textit{for some } M > 0$$

then in fact

$$f(\sigma + it) = \mathcal{O}(t^M) \quad \sigma \in [\sigma_1, \sigma_2]$$

uniformly in $\sigma \in [\sigma_1, \sigma_2]$.

Proof. We wish to use the maximum modulus principle, which states that a holomorphic map on a domain of $\mathbb{C}$ attains it maximum on the boundary. Define $B[a, b] = \overline{R[\sigma_1, \sigma_2]} \cap \{a \leq \mathrm{Im}(s) \leq b\}$ to be a rectangle. We start by noting that for fixed $[\sigma_1, \sigma_2]$, there is $\theta_0 = \theta(\sigma_1, \sigma_2)$ such that for polar-coordinates $s = re^{i\theta}$, θ, θ stays uniformly close to θ_0 for all $s \in B[a, \infty]$ only depending on a. Let $m > \alpha$ such that $\cos m\theta_0$ is close to -1. It suffices to proof the proposition for f replaced by $f(s)s^{-M}$. Fix $\varepsilon > 0$ and let $g_\varepsilon(s) = f(s)e^{\varepsilon s^m}$. By assumption on f, there exists t_0 such that

$$|g_\varepsilon(s)| \ll e^{t^\alpha + \varepsilon r^m \cos m\theta} \quad s \in B[t_0, \infty]$$

As $r \leq t$, $m > \alpha$, $\theta \sim \theta_0$, there is t_ε with

$$|g_\varepsilon(s)| \leq 1 \quad s \in B[t_\varepsilon, \infty]$$

Using the second growth assumption, we deduce that $|g_\varepsilon(s)|$ is bounded on the boundary of $B[t_0, t_\varepsilon]$, with implicit contants only depending on f. By the maximum modulus principle,

$$|f(s)| \ll e^{-\varepsilon r^m \cos m\theta}$$

for any $s \in B[t_0, t_\varepsilon]$. Letting $\varepsilon \to 0$ (and thus $t_\varepsilon \to \infty$), $|f(s)| \ll 1$ for any $s \in B[t_0, \infty]$, what was to be proven. □

Lemma 9.4 (The shifting contour argument)**.** *Let ϕ be smooth and compactly supported. The formula*

$$\phi(y) = \frac{1}{2\pi i} \int_{(c)} \Psi(s) y^{-s} ds.$$

is independent of (c), and holds for any c.

Proof. By absolute convergence, we may apply the dominated convergence theoreom to justify interchanging integral and differentiation to see that $\Psi(s)$ is holomorphic. From Lemma 9.2, $\Psi(s) = \mathcal{O}((1 + |s|)^{-k})$ for any k, $\int_{(c)} y^s \Psi(s) ds$ is absolutely convergent for any c. Since $y^s \Psi(s)$ is holomorphic, Cauchy's theorem states that

$$\int_{B[-T,T]} y^s \Psi(s) = 0.$$

We split the integral over $\int_{B[-T,T]}$ into 4 integrals. Applying the lemma again, the horizontal sides are dominated again by $\int_{\sigma_1}^{\sigma_2} y^t (1 + t + iT)^{-k} dt$ going to zero for k sufficiently large. Taking $T \to \infty$,

$$0 = \lim_T \int_{B[-T,T]} y^s \Psi(s) = \int_{(\sigma_1)} y^s \Psi(s) ds - \int_{(\sigma_2)} y^s \Psi(s) ds$$

□

10. Appendix: Counting with Non-abelieuan Harmonic Analysis

Let ρ be any unitary representation of G and λ denote the regular representation of G on $L^2(G)$. They extend to *representations of $L^1(G)$, so that $\lambda(f)$ simply acts by convolution on $L^2(G)$. An important notion is that of weak containment $\sigma < \rho$ of two representations. Say that $\sigma < \rho$ if $\|\sigma(f)\| \leq \|\rho(f)\|$ for any $f \in L^1(G)$. Weak containment can be rephrased in terms of matrix coefficents.

Theorem 10.1. *Let (ρ, H) be a unitary cyclic representation of G and $v \in H$. Then if $g \mapsto \langle \rho(g)v, v\rangle_H$ is in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$ then $\rho < \lambda$.*

Decompose $G = KP$ and $P = AN$. Let $\Xi(g) = \int_K \Delta(gk)^{-1/2} dk$ is the Harish-Chandra function and Δ is the modular function defined by $dg = dkdp = \Delta(a) dkdadn$.

Theorem 10.2 (CHH)**.** *Let (ρ, H) be a unitary cyclic representation of G and assume $\rho < \lambda$. Then for v, w K-eigenvectors in H*

$$|\langle \rho(g)v, w\rangle| \leq \Xi(g) \|v\| \|w\|$$

In terms convolution it has the following formulation. Here it becomes important to know that $\Xi \in L^q(G)$ for any $q > 2$.

Theorem 10.3 (Kunze-Stein, Cowling?). *Let $p \in [1,2)$ and $f \in L^1(G) \cap L^p(G)$. Then the operator bound of the convolution action of f on $L^2(G)$ satisfies*

$$\|f \star \varphi\|_2 \leq \|\Xi\|_q \|f\|_p \|\varphi\|_2$$

where q is the adjoint of p.

As remarked, this bound is true for any representation weakly contained in the regular representation. We call a representation tempered if its matrix coefficients are in $L^{2+\varepsilon}(G)$. For a representation for which knows L^q integrability for some $q > 2k$ we still have the statement of 10.2 for Ξ replaced by $\Xi^{1/k}$ [Corollary, CHH]. See Nevo, Gorodnik-Nevo for a more detailed discussion on this transfer principle. Call such a q the integrability exponent of representation

We see from Theorem 10.3 the following convolution bound for β_t the uniform probability measure supported on B_t, where we consider ρ the action on $L^2_0(G/\Gamma)$ known to have finite integrability exponent.

Theorem 10.4 (Gorodnik-Nevo). *For any $\psi \in L^2_0(G/\Gamma)$*

$$\|\beta_t \star \psi\|_2 \ll m_G(B_t)^{-\frac{1}{q}} \|\psi\|_2$$

A outline the duality principle for the counting problem. Let π be the projection of compactly supported funtions on G to G/Γ,

$$\pi_f(g\Gamma) = \sum_{\gamma \in \Gamma} f(g\gamma)$$

Then

$$|g\Gamma \cap B_t| = \pi_{\mathbb{1}_{B_t}}(g\Gamma) = \sum_{\gamma \in \Gamma} \mathbb{1}_{B_t}(g\gamma)$$

shall allow count $\Gamma \cap B_t$ for a family of sets $B_t \subset G$. We relaxed the counting problem to counting for arbitrary $g\Gamma$. For $g\Gamma$ close to Γ we find similar asymptotics. Call B_t Hoelder admissable (Def 1.2 GN) if

$$B^G_\varepsilon B_t B^G_\varepsilon \subset B_{t+c\varepsilon^a} \quad \text{and} \quad m_G(B_{t+\varepsilon}) \leq (1 + c\varepsilon^a) m_G(B_t).$$

Admissability is simply an assumption on the wellroundedness of the sets B_t. We see that dilated balls are Hoelder admissable, but note that the first condition does not allow for cones in G. One can allow oneself to such restrictive shapes by adding parameters on $B_{t_1,\ldots,t_n}$ coming from various group decompositions, see GN Chapter 8 for Cartan and HN for Iwasawa. We discuss proofs of wellroundedness in Section 3.1 and Section 3.4. In the context of counting saddle connections also NRW. For a general famility of functions f_t on G one wishes to have a bound of the form

$$\pi_{f_t}(g\Gamma) = (1 + \mathcal{O}(\beta(\varepsilon))\, \pi_{f_{t+\alpha(\varepsilon)}}(\Gamma)$$

for some positive decreasing functions α, β.

Back to $f_t = \mathbb{1}_{B_t}$ we may average against ϕ_ε a dirac approximation of the identity the G,

$$\langle \pi_{\mathbb{1}_{B_{t-\varepsilon}}}, \phi_\varepsilon \rangle_G \leq (1 + \mathcal{O}(\varepsilon))|\Gamma \cap B_t| \leq (1 + \mathcal{O}(\varepsilon)) \langle \pi_{\mathbb{1}_{B_{t+\varepsilon}}}, \phi_\varepsilon \rangle_G.$$

This viewpoint was iniated by Bartels, DRS, EM and is developed to great extend by GN. Since the inner product can be rewritten as non-normalized convolution $\beta_t \star \psi_\varepsilon$

$$\langle \pi_{f_t}, \phi_\varepsilon \rangle_G = \int_G \pi_{f_t}(g\Gamma)\phi_\varepsilon(g)dg = \sum_\gamma \int_G f_t(g)\phi_\varepsilon(g\gamma^{-1})dg = f_t \star \pi_{\phi_\varepsilon}(g),$$

whose variation has polynomial decay, from which by a Borel-Cantelli argument one gets almost all pointwise counting of $|g\Gamma \cap B_t|$. Homogenity implies everywhere counting. See NRW for a related discussion on a non homogeneous space.

Theorem 10.5 (Gorodnik-Nevo). *For admissable (defined slightly different) B_t,*

$$|\Gamma \cap B_t| = \frac{m_G(B_t)}{m_{G/\Gamma}(G/\Gamma)} + \mathcal{O}(m_G(B_t)^{-*})$$

Remark 10.6. One can improve the exponent in Theorem 10.5 if the sets are bi-K invariant.

11. Appendix: Eisensteinseries: Meromorphic continuation to the >1/2-strip

It is possible to avoid deep spectral inputs such as Theorem 2.4 and even give up apriori knowledge of the meromorphic continuation of $E(z,s)$ to the complex plane. We follow Truelsen [Uniform equidistribution] for this exposition.

We will use a different Casimir operator.

$$\mathcal{C}E_m(g,s) = (s(s-1) - \frac{5}{4}m^2)E_m(g,s),$$

for the Casimir operator $\mathcal{C}$ given in coordinates $g = n(x)a(y)k(\theta)$ by

$$\mathcal{C} = y^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) + y\frac{d^2}{dxd\theta} + \frac{5}{4}\frac{d^2}{d\theta^2}$$

It will also be important to use smooth truncations.

Definition 11.1 (Truncated Eisenstein Series [8])**.** Let $h : \mathbb{R}_+ \to \mathbb{R}$ with $h(y) = 0$ for $y < A$ and $h(y) = 1$ for $y > A + 1$.

$$F_m(g,s) = E_m(g,s) - h(y)y^s e(i\theta m)$$

Lemma 11.2.

$$\mathcal{C}F_m(g,s) = (s(s-1) - \frac{5}{4}m^2)F_m(g,s) - H_m(g,s)$$

where

$$H_m(g,s) = e(i\theta m)(h''(y)y^{s+2} + 2sh'(y)y^{s+1})$$

Proof. Take derivatives twice.[9] □

11.0.1. *Polynomial growth in L^2 and meromorphic continuation.*

Lemma 11.3. *Let $s = \sigma + it$ then*

$$\|H_m(g,s)\|_2^2 = \mathcal{O}(A^{2\sigma+2} + t^2A^{2\sigma}).^{10}$$

Proof. Observe that h',h'' are supported on $y \in [A, A+1]$ and uniformly bounded in the parameter A. We have $dg = \frac{1}{y^2}dydxd\theta$, so that

$$\|H_m(g,s)\|_2^2 \ll \sum_{\ell=1,2} |s^{2-\ell}|^2 \int_A^{A+1} |y^{s+\ell}|^2 \frac{dy}{y^2}$$

But each term individually is

$$\int_A^{A+1} |y^{s+\ell}|^2\frac{dy}{y^2} = \int_A^{A+1} y^{2\sigma+2\ell}\frac{dy}{y^2} = \frac{1}{2\sigma+2\ell-1}\left|(A+1)^{2\sigma+2\ell-1} - A^{2\sigma+2\ell-1}\right| = \mathcal{O}(A^{2\sigma+2\ell-2}).$$

□

Lemma 11.4. [11] *For $|t| \geq 1$,*

$$\|F_m(\cdot,s)\|_2 = \frac{1}{(2\sigma-1)|t|}\mathcal{O}\left(A^\sigma\sqrt{A^2+t^2}\right).$$

Proof. Let

$$R(s) = (\mathcal{C} - s(1-s) + \frac{5}{4}m^2)^{-1}$$

be the resolvent. The meromorphic continuation of $R(s)$ for $\sigma > \frac{1}{2}$ and holomorphicity of $H(g,s)$ implies in particular meromorphic continuation of $F_m(g,s)$, since

$$F_m(g,s) = R(s)(\mathcal{C} - s(s-1) + \frac{5}{4}m^2)F_m(g,s) = -R(s)H_m(g,s)$$

[8]In contrast to Kubota, here a smooth truncation is considered. This allows to conclude L^2 bounds with respect to the s parameter by a method of De Verdiere (Pseudo-Laplaciens II) and Goldfeld-Sarnak (Sums of Kloosterman sums) that has been applied in Truelsen and Risager-Rudnick for closely related counting resp. equidistribution problem.

[9]Note the misprint in the formula of H in both Truelsen and Risager-Rudnick. The factor s in the term $2sh'(y)y^{s+1}$ was forgotten. In particular, this will lead to a worse pointwise bound on $F_m(g,s)$ below.

[10]Note that Truelsen confusingly let $T = A$. Since H_m measures failure of square integrability, it is clear that it has to go to infinity as $A \to \infty$, which leads to useless bounds if $T = A$.

[11]Either finish proof for $|t| \leq 1$ citing the spectral properties of E_M in MS or restrict attention to $m = 0$.

The resolvent operator bound satisfies

$$\|R(s)\| = \frac{1}{\operatorname{dist}(s(s-1)-\frac{5}{4}m^2, \operatorname{spec}(\mathcal{C}))} \leq \frac{1}{\operatorname{Im}(s(1-s)-\frac{5}{4}m^2)} = \frac{1}{(2\sigma-1)|t|}.$$

Combining this with the previous lemma on the L^2 bound of H_m, we have for $|t| \geq 1$,

$$\|F_m(\cdot, s)\|_2 \leq \|R(s)\| \|(\mathcal{C} - s(1-s) + \frac{5}{4}m^2)F_m(\cdot, s)\|_2 = \frac{1}{(2\sigma-1)|t|}\mathcal{O}\left(A^\sigma \sqrt{A^2+t^2}\right).$$

□

Lemma 11.5. *It holds* [12]

$$\operatorname{Res}_{s=1} E_0(g,s) = \operatorname{Res}_{s=1} F_0(g,s)$$

Proof. The formula $F_m(g,s) = E_m(g,s) - h(y)y^s e(i\theta m)$ gives also analytic continuation of $E_m(g,s)$ for $\operatorname{Re} s > 1/2$, whose residue at 1 we know (to be the volume). But holomorphy of $h(y)y^s e(i\theta m)$ in this range implies that the residue of $F_m(g,s)$ agrees with that of $E_m(g,s)$. □

11.0.2. *Sobolev Embedding.*

Lemma 11.6.

$$\sup_{g\in\Omega} |F_m(g,s)| = \left(m^2/t + t\right) \mathcal{O}\left(A^\sigma \sqrt{A^2+t^2}\right)$$

Proof. We upgrade the L^2 bound to a pointwise bound by means of Sobolev inequality. For that we also need a bound for $\|\mathcal{C}F_m(\cdot,s)\|_2$, but clearly,

$$\|\mathcal{C}F_m(\cdot, s)\|_2 \leq \left|s(s-1) - \frac{5}{4}m^2\right| \|F_m\|_2 + \|H_m\|_2 = \left(\frac{m^2+t^2}{(2\sigma-1)|t|} + 1\right) \mathcal{O}\left(A^\sigma \sqrt{A^2+t^2}\right)$$

For any $\Omega \subset G/\Gamma$ compact,

$$\sup_{g\in\Omega} |F_m(g,s)| = \mathcal{O}(\|F_m(g,s)\|_2 + \|\Delta F_m(g,s)\|_2) = \left(m^2/t + t\right) \mathcal{O}\left(A^\sigma \sqrt{A^2+t^2}\right)$$

□

We now again use the approximative inequality 12 but by employment of the truncated integral representation

$$\sum_{\Gamma_\infty \backslash \Gamma} j_\gamma(g)^m \phi_U(y(\gamma g)^{-1}T^{-1}) - h(y)\phi_U(y^{-1}T^{-1})e(im\theta) = \frac{1}{2\pi i}\int_{(2)} F_m(g,s)\Psi_U(s)T^s ds.$$

The contour shifting argument, now powered by polynomial growth of F_m and sup-polynomial decay of Ψ_U by Lemma 9.2,

$$= \operatorname{Res}_{s=1}\left(F_m(g,s)\Psi_U(s)T^s\right) + \frac{1}{2\pi i}\int_{(\frac{1}{2}+\varepsilon)} F_m(g,s)\Psi_U(s)T^s ds$$

$$= \frac{\delta_{m,0}}{Vol(G/\Gamma)}\left(T + \mathcal{O}(T/U)(1+\mathcal{O}(1/A))\right) + \frac{1}{2\pi i}\int_{(\frac{1}{2}+\varepsilon)} F_m(g,s)\Psi_U(s)T^s ds$$

Using the pointwise bounds on $F_m(g,s)$ we bound the integral $\int_{(\frac{1}{2}+\varepsilon)} F_m(g,s)\Psi_U(s)T^s ds$. Since $F_m(g,s) = \mathcal{O}(\left(m^2/t+t\right)\mathcal{O}\left(A^\sigma\sqrt{A^2+t^2}\right) = \mathcal{O}(t^2 A^\sigma)$ for $|t| \to \infty$, we need to take $k = 2+\varepsilon$ in order for Ψ_U to counter this growth.

Finally we note that this leads inevitably to a worse error bound. Indeed, the average growth order from Theorem 2.7 is $|E(s)| = (|s|^{1/2})$, much sharper then our bound obtained here. This growth is known for the case of $\mathrm{SL}_2(\mathbb{Z})$ since $\phi(s)$ (appearing in the functional equation 8) is a ratio of Gamma and Zeta functions, see Terras Exercise 3.7.5. It is conjectured to be $|E(s)| = (|s|^\varepsilon)$ for any Γ.

[12] In Truelsen it is stated that

$$\operatorname{Res}_{s=1} E_0(g,s) = \operatorname{Res}_{s=1} F_0(g,s) + \mathcal{O}(1/A),$$

but as we argued, there's no error term of the type $\mathcal{O}(A^{-1})$ entering.

12. Appendix: Variance estimates of the Theta transform

One sees in Theorem 13.1 that the L^2-norm of an incomplete Eisenstein series $\|E(\psi,g)\|^2$ is an $L^2(R_{>0})$ will be controlled by the Mellin transform of ψ. We return to the Siegel-Veech transform which was defined for any $f:\mathbb{R}^2\to\mathbb{R}$ (and not just radially invariant). Below we describe an analouge of Theorem 13.1 for it, and serves as kind of Roger's formula, in particular leading to sharp variance bounds. We refer to Burton and the original Godement paper in which these ideas are employed. We shall follow Kelmer-Mohammadi for general Γ. We adapt to have the action of G on row vectors and take now a group level view point. We also aim to explain how Theorem 2.7 follows from the weight 0 case, using raising/lowering operators. The Iwasawa decomposition $G=NAK$. Let $P=NA$ the upper triangular group. Haar becomes $dg=e^{-t}dtdxdk$ coordinates for $g=n_xa_tk$ with $a_t=\operatorname{diag}\left(e^{t/2},e^{-t/2}\right)$, $k_\theta=\left[\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right]$, $n_x=\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right]$. Let $Q=MN$, $M=\{\pm1\}=K\cap A$ so that $Q\backslash G=M\backslash AK$. [13] Coordinates and normalization are given by,

$$\int_{Q\backslash G} f(g)dg=\int_{\mathbb{R}}\int_{M\backslash K} f(a_tk)e^{-t}dtdk \text{ and } \int_G f(g)dg=\int_{Q\backslash G}\int_Q f(qg)dqdg$$

We now look on the stabilizer $\Gamma_\infty=\Gamma\cap P$. For $f\in C_c^\infty(Q\backslash G)$ (which is Γ_∞-invariant) define the theta function

$$\Theta_f(g)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma g).$$

The following unfolding formula holds

Lemma 12.1. *(Lemma 2.1 KM) For any $F\in L^2(\Gamma\backslash G)$ and $f\in C_c^\infty(Q\backslash G)$*

$$\int_{\Gamma\backslash G}\Theta_f(g)F(g)dg=\int_{\Gamma_\infty\backslash G} f(g)F(g)dg$$

One may fold once more and use Q-invariance of f, so that an integral over $\Gamma_\infty\backslash Q$ in the right hand side corresponds to taking the zero'th Fourier coefficient with respect to the N-action:

$$\langle\Theta_f,F\rangle_{\Gamma\backslash G}=\langle f,F^\circ\rangle_{Q\backslash G} \tag{21}$$

The Adjoint operator of the Theta-transform is taking the zero Fourier coefficient. This reflects the orthogonality of the continuous spectrum containing the image of the incomplete Eisenstein transform with cusp forms.

Let $\phi_s\in C^\infty(Q\backslash G/K)$ be $\psi_s(na_tk)=e^{st}$. Let $E(g,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\psi_s(\gamma g)$ the standard Eisenstein series (with the substitition of variables $y=e^t$ to our original definition) and

$$E^\circ(s,g)=\int E(s,n_xg)dx$$

integrating over $\Gamma_\infty\backslash Q$, or in local variables, $\mathbb{R}/\mathcal{O}_\Gamma$, $\mathcal{O}_\Gamma=\{x\in\mathbb{R}:n_x\in\Gamma_\infty\}$ (the zero'ths fourier coefficient of $E(s,g)$) which satisfies

$$E^\circ(s,g)=\psi_s(g)+\mathcal{C}_\Gamma(s)\psi_{1-s}(g)$$

where $\mathcal{C}_\Gamma(s)=\phi(s)$ in the notation of the functional equation 8.

Let $\phi_m\in L^2(M\backslash K)$ denote the function $\phi_m(k_\theta)=e^{2im\theta}$, naturally seen as function on G. A function on G satisfying $\phi_m(gk)=\phi(g)\phi_m(k)$ is called a K-eigenfunction of weight m. Let $\psi_{s,m}(g)=\psi_s(g)\phi_m(g)$.

Let (h,e,f) be the standard $\mathfrak{sl}_2(\mathbb{R})$-tripel for the adjoint representation of G and define raising and lowering operators by

$$a^\pm=h\pm i(e+f)$$

sending weight m-eigenfunctions to $m\pm1$ eigenfunctions. We note here $a^\pm\phi_{s,m}=-2(s\pm m)\phi_{s,m\pm1}$ and the L^2-decomposition $L^2(M\backslash K)=\bigoplus_m L^2(M\backslash K,m)$.

Define the non-spherical Eisenstein series for $\phi\in L^2(M\backslash K,m)$ into K-eigenspaces of weight m,

$$E(\phi,s,g)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\psi_s(\gamma g)\phi(\gamma g).$$

Of course, these are precisely the weight m Eisenstein series from Section 2.2.

Then integrating over $\Gamma_\infty\backslash Q$, gives.

[13] In particular, one should think of $f:Q\backslash G\to\mathbb{R}$ as even function on $\mathbb{R}$. To reduce notation, one might just restrict to a torsion free sublattice, which always exists.

Proposition 12.2 (KM, Proposition 2.2)**.** *For $\phi \in L^2(M\backslash K, m)$,*

$$E^{\circ}(\phi, s, g) = (\psi_s(g) + P_m(s)\mathcal{C}(s)\psi_{1-s}(g))\,\phi(g)$$

and $P_m(s)$ defined in (2.11, KM).

We wish to again assume that Γ is tempered - i.e. $\mathcal{C}(s)$ has no exceptional poles besides 1. In the general case, these have to be added to the bound below. For arithmetic lattices, one can bound the poles in terms of the continues contribution. For non-arithmetic one may restrict attention to functions that allow separability of variables to again deduce such bounds, see KM, Theorem 3.

Proposition 12.3 (Proposition 2.3 KM)**.**

$$\|f\|_1^2 \ll \|\Theta_f\|_2^2 \ll \|f\|_2^2 + \|f\|_1^2$$

Here $\|\cdot\|_1, \|\cdot\|_2$ are with respect to Haar on $Q\backslash G$, i.e. $e^{-t}dtdk$.

Sketch of upper bound. An arbitrary function $f \in C_c^{\infty}(Q\backslash G)$ can be decomposed into K-eigenfunctions $f = \sum_m f_m$, where $f_m \in L^2(M\backslash K, m)$, that is, $f_{m,l}(ka) = \hat{f}_m(a)\phi_m(k)$ and

$$\hat{f}_m(a) = \int_K f(ak)\overline{\phi_m}dk$$

is the $m's$ Fourier coefficient of f. We have

$$\|\Theta_f\|_2^2 = \sum_m \|\Theta_{f_m}\|_2^2 \tag{22}$$

Let $f(g) = f(a_tk) = v(t)\phi(k)$ for $v_t \in C_c^{\infty}(0,\infty)$ and $\phi = \phi_m$ then

$$f(g) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{v}(r - i\sigma)\psi_{\sigma+ir}(g)\phi(g)dr$$

for $\hat{v}(r) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} v(t)e^{-irt}dt$, $r \in \mathbb{C}$. We note that the variable change from y to e^t caused to consider the Fourier transform instead of the Mellin transform. This leads to the integral representation of Theorem 2.3, formula 9 for these non-spherical Eisenstein series:

$$\Theta_f(g) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{v}(r - i\sigma)E(\phi, \sigma + ir, g)dr \tag{23}$$

Integrating over $\Gamma_\infty\backslash Q$, we get from Proposition 12.2 with a contour shift argument that

$$\int_{\mathcal{F}_{\mathcal{O}_\Gamma}} \Theta_f(nak)dn$$

(24)

$$= c\phi(k)\left(\int_{\mathbb{R}} \hat{v}(r - i/2)\phi_{1/2+ir}dr + \int_{\mathbb{R}} \hat{v}(r - i/2)\mathcal{C}(\frac{1}{2} + ir)P_m(1/2 + ir)\psi_{1/2-ir}dr + 2\pi P_m(1)\hat{v}(-i)\right)$$

where the last term is the Residue at $s = 1$(see equation (2.15) in KM) (that vanishes if $m \neq 0$.

We now use equation 21, which for $F = \Theta_f$ reads

$$\|\Theta_f\|_2^2 = c'\int_K\int_{\mathbb{R}} \overline{f(a_tk)}e^{-t}\int_{\mathcal{F}_{\mathcal{O}_\Gamma}} \Theta_f(n_xa_tk)dxdtdk$$

Apply this for $f = v\phi$ with the representation of 24 to get

$$\|\Theta_f\|_2^2 = c''\int_{\mathbb{R}} |\hat{v}(r - \frac{i}{2})|^2dr + \int_{\mathbb{R}} \hat{v}(r - \frac{i}{2})\overline{\hat{v}(-r - \frac{i}{2})}\mathcal{C}(\frac{1}{2} + ir)P_m(\frac{1}{2} + ir)dr + 2\pi P_m(1)|\hat{v}(-i)|^2. \tag{25}$$

Sum over m and apply equation 22 to bound $\|\Theta_f\|_2^2$ for general f.

For each summand m we bound as follows. We note that the $\|f\|_1$ term only enters through $m = 0$. For the first term we have Plancherel: $\int_{\mathbb{R}} |\hat{v}(r - \frac{i}{2})|^2dr = \int_{\mathbb{R}} |v(t)|^2e^{-t}dt = \|f\|_2^2$. For the second term, we call that P_m and $\mathcal{C}$ are unitary on the critical line, so that Cauchy-Schwartz reduces this summand to the first term. For the third summand, i.e. $m = 0$, we simply have $2\pi|\hat{v}(-i)|^2 = |\int_{\mathbb{R}} v(t)e^{-t}dt|^2 = \|f\|_1^2$. $\square$

13. Appendix: Mass-Selberg-Relations and their spectral consequences

13.1. **Eisenstein Series III - Spectral discussion.** We continue with a discussion on the spectral theory of the incomplete Eisenstein series (Theorem 2.3). Introduce the Eisenstein transform of a function $f \in C^\infty(\mathbb{R} > 0)_0$

$$(Ef)(z) = \frac{1}{4\pi}\int_0^\infty f(r)E(z,\frac{1}{2}+ir)$$

Theorem 13.1 (Iwaniec Proposition 7.1)**.**

$$\langle Ef, Eg\rangle = \frac{1}{2\pi}\int f(r)g(r)dr$$

This theorem relies on the Maass-Selberg relations of truncated Eisenstein series defined by

$$\tilde{E}^Y(z,s) = E(z,s) - \operatorname{Im}(z)^s - \phi(s)\operatorname{Im}(z)^{1-s}$$

for $z \in F(Y) = \{z : \operatorname{Im} z > Y\}$ and $\tilde{E}^Y(z,s) = E(z,s)$ if $z \in F(Y)^c$. satisfying an exact formula for the expression $\langle \tilde{E}^Y(\cdot,s_1), \tilde{E}^Y(\cdot,s_2)\rangle$ (see Iwaniec Proposition 6.8). For $s = \frac{1}{2}+it$, $t \neq 0$ one obtains in particular (equation 6.35 Iwaniec) that

$$\langle \tilde{E}^Y(\cdot,s), \tilde{E}^Y(\cdot,s)\rangle = \frac{1}{2s-1}(\phi(1-s)Y^{2s-1} - \phi(s)Y^{1-2s}) + 2\log Y - \frac{\phi'}{\phi}(s)$$

We note that ϕ is unitary on the critical line, which implies that the L^2-norm of $\tilde{E}^Y(\cdot,s)$ is essentially governed by $2\log Y - \frac{\phi'}{\phi}(s)$. Define $M_\Gamma(T) = \frac{1}{4\pi}\int_{-T}^T -\frac{\phi'}{\phi}(\frac{1}{2}+it)dt$ for later use.

Theorem 2.4 is proven by use of Theorem 13.1. Let us first note that the dependence on z in Theorem 2.4 can be made explicit,

$$\int_{-T}^T |E(z,\frac{1}{2}+it)|^2 dt \ll T^2 + Ty_\Gamma(z)$$

where $y_\Gamma(z)$ is the *height* of z.

Proof of Theorem 1.3, following Iwaniec[14] . We only proof it for z fixed, without the refinement $\mathcal{O}(Ty_\Gamma(z))$. The rough idea is to apply Theorem 13.1 for $f(r) = g(r) = \mathbb{1}_{[-T,T]}(r)E(z,\frac{1}{2}+ir)$. To make things work, one actually takes $g = h(r)E(z,\frac{1}{2}+ir)$, where h comes from the following: Recall the following fact about the spectral theory of the Laplacian on $\mathbb{H}$: Any eigenfunction of eigenvalue $\lambda = s(1-s)$, $s = \frac{1}{2}+it$, $t \in \mathbb{C}$ is an eigenfunction for integral operators L_k of point pair invariants $k(z,w)$ (i.e. $k(gz,gw) = k(z,w)$), where,

$$L_k f(w) = \int_{\mathbb{H}} k(z,w)f(z)dz.$$

Thus if $\Delta f = \lambda f$, there is $h_k(s)$ such that $L_k f = h_k(t)f$. h is called Selberg-Harish-Chandra-transform. Since h is independent of f, we may take $f = y^s$ to get the integral representation $\int_{\mathbb{H}} k(z,w)y(z)^s dz = h(t)y^s$. We let $k(z,w) = \mathbb{1}_{B_\delta(w)}(z)$ where $B_\delta(w)$ is the ball of hyperbolic *volume* δ. δ is small and will be taken to be $1/T^2$. For $s = 0$, we find in particular that $h(i/2) = \int_{\mathbb{H}} k(z,w)dz = \delta$. A calculation (to check) shows that $\delta \ll h(t) \ll \delta$ for any $|s| \ll \delta^{-1/2}$.

Let $f(z) = \sum_{\gamma\in\Gamma} k(\gamma w, z)$ the automorphic kernel associated to k. We find

$$\|f\|^2_{L^2(\mathbb{H}\backslash\Gamma)} = \sum_{\gamma',\gamma}\int_{\mathbb{H}\backslash\Gamma} k(\gamma z,w)k(\gamma' z,w)d\mu(z) = \sum_\gamma \int_{\mathbb{H}} k(z,w)k(\gamma z,w)dz$$

We note that the integrant is positive only if both w and γw is inside the δ ball $B_\delta(z)$. For δ sufficiently small, this requires $\gamma = id$, thus

$$\int_{\mathbb{H}\backslash\Gamma} k(z,w)k(\gamma z,w)d\mu(z) = \int_{\mathbb{H}} k(z,w)^2 dz = \delta.$$

As mentioned earlier, we now take $g = h(r)E(z,\frac{1}{2}+ir)$, use first unfolding then interchange integrals to see

$$\langle F, Eg\rangle = \int_{\mathbb{H}} k(z,w)Eg(z)dz = \int_0^\infty g(r)(L_k E(\cdot,\frac{1}{2}+ir))(w)dr$$

so Eg is the orthogonal projection of F to Eg. Since E is an eigenfunction,

$$\int_0^\infty g(r)(L_k E(\cdot,\frac{1}{2}+ir))(w)dr = \int_0^\infty g(r)h(r)E(w,\frac{1}{2}+ir)dr = \|g\|^2 = \|Eg\|^2$$

By Cauchy-Schwartz therefore $\|Eg\| \leq \|F\|$. Finally,

$$\int_{-T}^{T} |E(z, \frac{1}{2} + ir)|^2 dr \ll \int_{-T}^{T} \left|\frac{h(r)}{\delta}\right|^2 |E(z, \frac{1}{2} + it)|^2 dr \leq \delta^{-2}\|hE\|^2 = \delta^{-2}\|Eg\|^2 \leq \delta^{-2}\|F\|^2 = \delta^{-1} = T^2$$

□

The Eisenstein series admit a Fourier expansion (Iwaniec, equation 8.2) $E(z,s) = y^s + \phi(s)y^{1-s} + \sum_{n\neq 0} \phi(n,s)W_s(nz)$ which by Theorem 2.4 satisfies $\int_{-T}^{T} \sum_{n\neq 0} |\phi(n,s)W_s(nz)|^2 dt \ll T^2 + yT$. We shall see this fact again in Theorem 14.4.

Using these estimates in combination with Selberg's trace formula one can deduce (equation 10.13 Iwaniec) $M_\Gamma(T) \ll T^2$ and therefore

Theorem 13.2.

$$\int_{-T}^{T} \|\tilde{E}^Y(\cdot, \frac{1}{2} + it)\|^2 \ll T^2$$

Remark 13.3. This estimate is in fact valid for any $\operatorname{Re} s \geq \frac{1}{2}$, see Sarnak, Horocycles.

This is the spectral input for Sarnak's proof of the effective equidistribution of horocycles. We note that by the strategy of Duke-Rudnick-Sarnak and Eskin-Mcmullen, one can obtain a polynomial error term count by employing precicely this fact. We come back to this in Chapter 3. It is possible to shorten this dynamical argument of counting as we show in **??**. We see that the argument boils down to the same reason as in the proof of Theorem 2.8.

14. Appendix: Previous work on closed Horocycles

14.1. **Proof of equidistribution of low horocycles.** We discuss the two approaches to Theorem 3.3. Let again $\nu_y = \nu_t = a_{t*}m_Y$ orbit measure of a closed horocycle of length $t = 1/y^{\frac{1}{2}}$. The Rankin-Selberg method has been suggested by Zagier [Eisenstein Series and Rieman-Zeta-Function] for $\mathrm{SL}_2(\mathbb{Z})$. This has been carried through by Sarnak [Horocycles] for general Γ. We will take the Mellin transform $E(\nu)(s)$ of ν_y. As in equation 7 for the counting function we shall get an integral representation involving integration against $E_m(g,s)$ defined in 16. The second argument holds for for K-invariant functions, but strengthended to hold on short segments and quantifies a result of Heyhal. We finish on a discussion on mixing.

14.2. **Zagier-Sarnak's argument.** In local coordinates (x,y,θ), $\nu_y(f) = \int_0^1 f(x,y,0)dx$. Now take (measure valued) mellin transform

$$E(\nu_\bullet)(s) = \int_0^\infty \nu_y y^{-s-2} dy$$

Let $f(x,y,\theta) = \sum \hat{f}(n,z)e^{in\theta}$ the Fourier decomposition along K. Its fourier coefficient, as function on $\mathbb{H} = G/K$ that is function of weight n with respect to Γ satisfying $\hat{f}(n,\gamma z) = e^{i2n\arg(cz+d)}\hat{f}(n,z)$. Integrating f against $E(\nu_\bullet)(s)$ gives

$$E(\nu_\bullet)(s)(f) = \sum_n \int_{\Gamma\backslash\mathbb{H}} \hat{f}(n,z)E_n(z,s)dz$$

and by Mellin inversion $\nu_y = \frac{1}{2\pi}\int_{(c)} E(\nu_\bullet)(s)y^{-s}ds$ for any $c > 1$ and so

$$\nu_y(f) = \frac{1}{2\pi}\int_{(c)} \left(\sum_n \int_{\Gamma\backslash\mathbb{H}} \hat{f}(n,z)E_n(z,s)y^s dz\right) ds$$

To show that $\left(\sum_n \int_{\Gamma\backslash\mathbb{H}} \hat{f}(n,z)E_n(z,s)y^{-s}dz\right)$ is absolutely integrable, one argues by partial integration with the Laplacian, see equation 20. The spectral bound of the Eisenstein series comes again from Theorem 13.2. To allow for truncated Eisenseries, one uses that f is compactly supported (or more generally, with fast decay in the cusp). Shifting to c to the critical line we see the $t^{-1} = y^{1/2}$ remainder term appearing, the Theorem follows after showing that the residue is $m_{\Gamma\backslash G}(f)$ (the integral of f against trivial eigenfunction).

Remark 14.1. Close the cycle and add discussion how counting implies equidistribution? Probably based on 21.

14.3. **Strombergssen.** Let $D = -y^2(\partial_x^2+\partial_y^2)$ with eigenfunctions $\{\phi_i\}$ (but no cusp forms among them) and $\mathcal{S}(f) = \|f\|_2 + \|Df\|_2$.

Theorem 14.2. *Let $f \in L^2(\Gamma\backslash\mathbb{H})$ be smooth and $y^{\frac{1}{2}} \leq b-a \leq 1$.*

$$\frac{1}{b-a}\int_a^b f(x+iy)dx = m_{\Gamma\backslash\mathbb{H}}(f) + \mathcal{O}\left(\mathcal{S}(f)\frac{y^{\frac{1}{2}-\varepsilon}}{b-a}\right)$$

Proof. There is a spectral decomposition (with continuous part coming from 11 in Theorem 2.3).

$$f(z) = \sum d_m\phi_m(z) + \int g(R)E(z,\frac{1}{2}+iR)dR$$

and

Theorem 14.3.

$$E(z,\frac{1}{2}+iR) = y^{\frac{1}{2}+iR} + \phi(\frac{1}{2}+iR)y^{\frac{1}{2}-iR} + \sum_{n\neq 0} c_n(R)\sqrt{y}K_{iR}(2\pi|n|y)e(nx)$$

so that

$$\frac{1}{b-a}\int_a^b E(z,\frac{1}{2}+iR) = y^{\frac{1}{2}+iR} + \phi(\frac{1}{2}+iR)y^{\frac{1}{2}-iR} + \frac{1}{b-a}\sum_{n\neq 0} c_n\sqrt{y}K_{iR}(2\pi|n|y)\frac{e(nb)-e(na)}{2\pi in}.$$

and is Theorem 12.2 for $[a,b]=[0,1]$.

Theorem 14.4. *(Proposition 4.1 Strombergssen) The Fourier coefficients c_n satisfy*

$$\sum_{n<N}|c_n|^2 = \mathcal{O}\left(e^{\pi R}(N+R))\omega(R) + \log(\frac{2N}{R+1}+R)\right)$$

uniformly for any N and R where $\omega(R) = -\frac{\phi'}{\phi}(\frac{1}{2}+iR)$ from Section 13.1.

The spectral input is Theorem 13.2.
Combined with known asymptotics of K_{iR} [4.15 Strombergssen]

$$K_{iR}(y) = \mathcal{O}\left(e^{-\frac{\pi}{2}R}(R+1)^{-1/3+\varepsilon}y^{-\varepsilon}\min(1, e^{\frac{\pi}{2}R-y})\right) \tag{26}$$

one has

Theorem 14.5 (Proposition 4.2 Strombergssen)**.**

$$\frac{1}{b-a}\int_a^b E(z,\frac{1}{2}+iR) = \mathcal{O}(y^{1/2-\varepsilon})\left(1+\frac{(1+R)^{1/6+\varepsilon}\sqrt{\omega(R)}}{b-a}\right)$$

To bound therefore the continuous part of $\frac{1}{b-a}\int_a^b f(x+iy)dx$, we integrate the last formula against $g(R)$, which is bounded by the arguments entering Theorem 2.8. One obtains the integrability of $g(R)$ from nowing that Df has also a spectral decomposition with fourier coefficient a function of $g(R)$ times a polynomial in R. The discrete part is treated similarly. □

14.4. **Adaption to Counting.** Of course, the proof of Theorem 14.2 in particular implies to an Incomplete Eisenstein transform. We pick up at trying to bound 14, $\int_{(\frac{1}{2})} E(g,s)\Psi_U(s)T^s ds$ in terms of U. In view of Theorem 14.3 we wish to give a bound for $\sum_{n\neq 0} c_n(R)\sqrt{y}K_{iR}(2\pi|n|y)e(nx)$ in s - which can be collected from equation 26 and Theorem 14.4. One obtains $\mathcal{O}(R^\kappa\sqrt{\omega(R)})$ for some $\kappa \geq 0$. With the super-polynomial decay of $\Psi_U(s)$ from Lemma 9.2 one can proceed as in Theorem 2.8.

14.5. **Mixing.** Equidistribituion of horocycles on $\Gamma\backslash G$ can also be obtained by mixing. This method also allows to consider pieces of horocycles on cocompact lattices or closed horocycles [Venkatesh] and restriction to short segments [Kelmer Kontorovich, Shears].

14.6. **non-closed orbit on non-uniform lattices.** Reference: Strombergssen [Deviation], Flaminio-Forni

14.7. **Epsilon discripancy.** Strombergssen losses an ϵ. FF $\log^2$

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