PERSONAL TRANSLATION OF JEAN-FRANÇOIS QUINT'S BOURBAKI NOTES ON "RIGIDITY OF $\mathrm{SL}_2(\mathbb{R})$-ORBITS IN THE MODULI SPACES OF FLAT SURFACES, AFTER ESKIN, MIRZAKHANI AND MOHAMMADI"

JEAN-FRANÇOIS QUINT. TRANSLATION BY RENE RUHR

Contents

Introduction

Let g be an integer ≥ 1 and $\alpha = (\alpha_1, \dots, \alpha_n)$ a partition of the integer $2g-2$. To this combinatorial data, we associate a stratum of Abelian differentials $\mathcal{H}(\alpha)$. The elements of $\mathcal{H}(\alpha)$ are isomorphism classes of pairs (M, ω) where M is a Riemann surface of genus g and ω is a holomorphic 1-form on M admitting exactly n zeros of multiplicities $\alpha_1, \dots, \alpha_n$. The space $\mathcal{H}(\alpha)$ is equipped with an analytic complex affine flat structure and a group action of the group $\mathrm{GL}_2(\mathbb{R})^+$ of real square matrices of size 2 and determinant > 0. In a series of recent works, Eskin and Mirzakhani and Eskin, Mirzakhani and Mohammadi showed that the orbit closures of the $\mathrm{GL}_2(\mathbb{R})^+$-orbits in $\mathcal{H}(\alpha)$ are affine complex subvarieties, thus answering affirmatively a conjecture of McMullen.

In this talk, after having described more precisely the structure of the space $\mathcal{H}(\alpha)$, I will give some indications on the proof of this theorem. I will focus essentially on the main result of [11], which uses notions of ergodic theory.

I sought the help of many colleagues during the preparation of this lecture. I would like to thank Christophe Bavard, Sylvain Crovisier, Vincent Koziarz and Carlos Matheus. I am particularly grateful to Duc-Manh N'Guyen, who answered patiently to my initial questions on translation surfaces, Yves Benoist, with whom we have discussed these topics extensively, and Alex Eskin who explained to me some particularly delicate points of the proofs. Finally, I thank Viviane Le Dret who has considerably improved the presentation of this text by removing many typos.

1. Strata of Abelian Differentials

Throughout this exposition, we fix an integer $g \geq 1$ and a compact orientable topological surface S of genus g. We will define translation structures with conic singularities on S and the moduli space of these structures. These spaces are equipped with a $\mathrm{GL}_2(\mathbb{R})^+$ group action.

1.1. Flat structures and holomorphic forms.

A flat chart with a conical singularity of S is a triple (U, φ, ψ) where U is an open set in S, φ is a homeomorphism from U to an open subset of $\mathbb{C}$ containing 0 and $\psi : U \to \mathbb{C}$ is the map $x \mapsto \varphi(x)^{\alpha+1}$, where α is an integer ≥ 0, uniquely determined by φ and ψ.

Two flat charts with conical singularity (U_1, φ_1, ψ_1) and (U_2, φ_2, ψ_2) are said to be compatible by translations if there exists c in $\mathbb{C}$ for which, for all x in $U_1 \cap U_2$, one has $\psi_2(x) = \psi_1(x) + c$. In particular, in this case, the transition map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a holomorphic function. A translation atlas with conical singularities on S is a collection $\mathcal{A}$ of flat charts with conical singularities, all of which compatible by translation, which is maximal for inclusion and such that $\bigcup_{(U,\varphi,\psi)} U = S$. Given such an atlas, we say that the pair $(S, \mathcal{A})$ is a translation surface with conical singularities. A singularity of $(S, \mathcal{A})$ is an element x of S such that there exists a map (U, φ, ψ) of $\mathcal{A}$ with $x \in U$, $\varphi(x) = 0$, $\alpha \geq 1$

where α is the integer such that $\psi = \varphi^{\alpha+1}$. We say that α is the order of the singularity x. The set of singularities of $(S, \mathcal{A})$ is finite.

Since the transition maps of $\mathcal{A}$ are holomorphic, the translation structure induces on S a Riemann surface structure. In addition, S is equipped with the holomorphic 1-form ω such that, on a chart (U, φ, ψ) of $\mathcal{A}$, we have $\omega = d\psi$. According to the theorem of Riemann-Roch, if ω has n zeros, of multiplicity $\alpha_1,\dots, \alpha_n$, one has $\alpha_1 + \cdots + \alpha_n = 2g - 2$. In other words, if $x_1, \dots, x_n$ are the singularities of $(S, \mathcal{A})$ and if $\alpha_1, \dots, \alpha_n$ are the orders of $x_1, \dots, x_n$, one has $\alpha_1 + \cdots + \alpha_n = 2g - 2$.

Conversely, the data of a Riemann surface structure on S and a non-zero holomorphic 1-form on it determines a translation surface structure with conical singularities on S.

Remark 1.1. It is probably simpler to define a translation surface structure as Riemann surface structure and a non-vanishing holomorphic 1-form. The heavy formalism of translation atlases that we have just introduced shall show its purpose when we define a $\mathrm{GL}_2(\mathbb{R})$ on the set of all flat surfaces.

1.2. **The space of translation surfaces.** Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a sequence of integers ≥ 1 such that $\alpha_1 + \cdots + \alpha_n = 2g - 2$. We denote by $\mathcal{P}(\alpha)$ the set of translation atlases with conical singularities of order $\alpha_1, \dots, \alpha_n$ on S. Note that the data of such an atlas determines an orientation of S. We fix once and for all an orientation of S and we denote $\mathcal{P}^+(\alpha) \subset \mathcal{P}(\alpha)$ the set of translation structures compatible with this fixed orientation.

Let $\mathcal{G}$ denote the group of homeomorphisms of S which preserve the orientation and $\mathcal{G}^\circ \subset \mathcal{G}$ the group of homeomorphisms isotopic to the identity, so that $\Gamma = \mathcal{G}/\mathcal{G}^\circ$ is the mapping class group of S. The group $\mathcal{G}$ acts naturally on $\mathcal{P}(\alpha)$ (and preserves $\mathcal{P}^+(\alpha)$): for g in $\mathcal{G}$ and $\mathcal{A}$ in $\mathcal{P}(\alpha)$, we write $g\mathcal{A}$ for the atlas whose charts are $(gU, \varphi \circ g^{-1}, \psi \circ g^{-1})$ where (U, φ, ψ) is a chart of $\mathcal{A}$.

We define the space $\mathcal{H}(\alpha)$ as quotient $\mathcal{P}^+(\alpha)$ by $\mathcal{G}$. We also denote by $\tilde{\mathcal{H}}(\alpha)$ the quotient $\mathcal{P}^+(\alpha)$ by $\mathcal{G}^\circ$, so that $\mathcal{H}(\alpha) = \Gamma\backslash\tilde{\mathcal{H}}(\alpha)$. We say that $\mathcal{H}(\alpha)$ is a stratum of Abelian differentials.

1.3. **The case of genus 1.** Suppose that $g = 1$, so that S is a torus and that α is necessarily empty partition. Choose a universal cover $\tilde{S} \to S$ of S of the covering group $\Lambda \simeq \mathbb{Z}^2$. The data of a flat structure on S turns into data of a developing map, which here is a diffeomorphism $D : \tilde{S} \to \mathbb{C}$ and a holonomy morphism (of discrete and cocompact image) $h : \Lambda \to \mathbb{C}$ such that for all $g \in \Lambda$ and all $x \in \tilde{S}$ we have $D(gx) = h(g) + D(x)$ (more exactly, the map D is determined up to translation by a constant element of $\mathbb{C}$ from the flat structure).

There is a natural isomorphism available between the cohomology groups $H^2(S, \mathbb{Z})$ and $H^2(\Lambda, \mathbb{Z})$ and the orientation of S determines a generator c of this cyclic group. Fix two generators g_1 and g_2 of Λ, so that $c(g_1, g_2) > 0$.

The data of the holonomy homomorphism completely characterises the flat structure up to isotopy. More precisely, the space $\tilde{\mathcal{H}}(\emptyset)$ is identified with the space of homomorphisms $h : \Lambda \to \mathbb{C}$ of discrete and cocompact image and such that the pair $(h(g_1), h(g_2))$ is positively oriented in $\mathbb{C}$. In other words, through this choice of generators, the space $\tilde{\mathcal{H}}(\emptyset)$ is identified with the set of tuples (z_1, z_2) in $\mathbb{C} \times \mathbb{C}$ with $\mathrm{Im}(\overline{z}_1 z_2) > 0$.

The group Γ is identified with $\mathrm{SL}_2(\mathbb{Z})$. One verifies that it acts on $\tilde{\mathcal{H}}(\emptyset)$ so that, if $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is in $\mathrm{SL}_2(\mathbb{Z})$ and (z_1, z_2) is in $\tilde{\mathcal{H}}(\emptyset)$, one has

$$\gamma(z_1, z_2) = (dz_1 - cz_2, -bz_1 + az_2).$$

Identify $\mathbb{C}$ with $\mathbb{R}^2$ by equipping $\mathbb{C}$ with the base $(1, i)$ of $\mathbb{C}$. Then we can see that $\tilde{\mathcal{H}}(\emptyset)$ is the space of oriented bases of $\mathbb{R}^2$. Let $\mathrm{GL}_2(\mathbb{R})^+$ the group of square matrices of size 2 and positive determinant. The group $\mathrm{GL}_2(\mathbb{R})^+$ acts on $\tilde{\mathcal{H}}(\emptyset)$ so that

$$g(z_1, z_2) = (gz_1, gz_2)$$

for $g \in \mathrm{GL}_2(\mathbb{R})^+$ and (z_1, z_2) in $\tilde{\mathcal{H}}(\emptyset)$. This action is simply transitive, the choice of the point $(1, i)$ in $\tilde{\mathcal{H}}(\emptyset)$ allows to see this set as a copy of $\mathrm{GL}_2(\mathbb{R})^+$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on this space is now read as the action on the right on $\mathrm{GL}_2(\mathbb{R})^+$, so that we can identify $\mathcal{H}(\emptyset)$ with the quotient $\mathrm{GL}_2(\mathbb{R})^+ / \mathrm{SL}_2(\mathbb{Z})$.

1.4. **The group action of** $\mathrm{GL}_2(\mathbb{R})^+$**.** Suppose again that g is any integer ≥ 1. We will now describe a $\mathrm{GL}_2(\mathbb{R})^+$ group action on $\mathcal{H}(\alpha)$ which generalises the one that has been introduced above when $g = 1$.

For that, let's start by trying to build a $\mathrm{GL}_2(\mathbb{R})^+$ action on the set $\mathcal{C}$ of flat charts with conical singularity of S. For (U, φ, ψ) in $\mathcal{C}$ and g in $\mathrm{GL}_2(\mathbb{R})^+$, we want $g(U, \varphi, \psi)$ to be of the form $(U, \varphi', g\psi)$ (where $g\psi$ is the function $x \mapsto g\psi(x)$ on U). If $\psi = \varphi$, we can put $\varphi' = \varphi$. On the other hand, if $\psi = \varphi^{\alpha+1}$ with $\alpha \geq 1$ (that is, if the chart contains a singularity), the function φ' must satisfy the equation $(\varphi')^{\alpha+1} = g\varphi$ and this equation has at least $\alpha + 1$ solutions. To remove this ambiguity, we will replace the group $\mathrm{GL}_2(\mathbb{R})^+$ by its universal cover.

Recall that $\mathrm{GL}_2(\mathbb{R})^+$ is connected and retracts continuously on $\mathrm{SO}(2)$. In particular, its fundamental group is isomorphic to $\mathbb{Z}$. Let us fix a universal covering $\pi : \widetilde{\mathrm{GL}_2(\mathbb{R})^+} \to \mathrm{GL}_2(\mathbb{R})^+$ with kernel Z. The continuous homomorphism of groups

$$\theta : \mathbb{R} \to \mathrm{SO}(2), t \mapsto \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix}$$

gives rise to a continuous group homomorphism $\tilde{\theta} : \mathbb{R} \to \widetilde{\mathrm{GL}_2(\mathbb{R})^+}$ and we have $\tilde{\theta}(\mathbb{Z}) = Z$. We put $c = \tilde{\theta}(1)$: this is the generator of Z[1] associated to the natural orientation of $\mathbb{C}$.

Elementary reasoning from the theory of covering spaces makes it possible to establish the

Lemma 1.2. *Let α be a natural integer. There is a continuous action by homeomorphisms of $\widetilde{\mathrm{GL}_2(\mathbb{R})^+}$ on $\mathbb{C}$, denoted by $(g, z) \mapsto g \cdot_\alpha z$, such that for all g in $\widetilde{\mathrm{GL}_2(\mathbb{R})^+}$ and z in $\mathbb{C}$ we have*

$$(g \cdot_\alpha z)^{\alpha+1} = \pi(g)(z^{\alpha+1})$$

(where, in the second term, $\mathrm{GL}_2(\mathbb{R})^+$ acts on $\mathbb{C} \simeq \mathbb{R}^2$ by the standard linear action).

For g in $\widetilde{\mathrm{GL}_2(\mathbb{R})^+}$, the map $z \mapsto g \cdot_\alpha z$ is homogeneous of degree 1. For all z in $\mathbb{C}$, one has

$$c \cdot_\alpha z = \exp\left(\frac{2i\pi}{\alpha + 1}\right) z. \tag{1}$$

Remark 1.3. Let $\alpha \geq 1$. Then for g in $\widetilde{\mathrm{GL}_2(\mathbb{R})^+}$, the map $z \mapsto g \cdot_\alpha z$ is $\mathcal{C}^\infty$ on $\mathbb{C} \setminus \{0\}$, but it is not differentiable in 0 since $\pi(g) \notin \mathbb{R}_+^* \mathrm{SO}(2)$. Indeed, as this map is homogeneous of degree 1, if it was differentiable in 0, it would be linear.

[1] typo. Original: c'est le generateur de $\mathbb{Z}$ associe a l'orientation naturelle de $\mathbb{C}$

The definition of the natural action of $\widetilde{\mathrm{GL}_2(\mathbb{R})}^+$ on the set $\mathcal{C}$ of flat charts with conical singularity now becomes clear: for g in $\widetilde{\mathrm{GL}_2(\mathbb{R})}^+$ and (U, φ, ψ) in $\mathcal{C}$, we put $g(U, \varphi, \psi) = (U, g \cdot_\alpha \varphi, g\psi)^2$ where α is such that $\psi = \varphi^{\alpha+1}$. We easily check that this action preserves the compatibility relations among the charts. In particular, if we fix a partition $\alpha = (\alpha_1, \ldots, \alpha_n)$ of integers ≥ 1 with $\alpha_1 + \cdots + \alpha_n = 2g - 2$, the action of $\widetilde{\mathrm{GL}_2(\mathbb{R})}^+$ on $\mathcal{C}$ induces an action on the space $\mathcal{P}^+(\alpha)$ of atlases compatible with the orientation and singularities of order $(\alpha_1, \ldots, \alpha_n)$. In conclusion, we have proved the

Lemma 1.4. *The group Z acts trivially on $\mathcal{P}^+(\alpha)$, so that the action of $\widetilde{\mathrm{GL}_2(\mathbb{R})}^+$ on this set factorises through an action of $\mathrm{GL}_2(\mathbb{R})^+$.*

Proof. Let $\mathcal{A}$ in $\mathcal{P}^+(\alpha)$ and we show that $c\mathcal{A} = \mathcal{A}$. For (U, φ, ψ) in $\mathcal{A}$, let β be the integer such that $\psi = \varphi^{\beta+1}$. According to ((1)), we have $c(U, \varphi, \psi) = (U, \exp\left(\frac{2i\pi}{\beta+1}\right)\varphi, \psi)$, and hence $c(U, \varphi, \psi)$ is compatible with all charts from $\mathcal{A}$. By maximality of the atlas $\mathcal{A}$, we have $c\mathcal{A} = \mathcal{A}$. □

Remark 1.5. Let $\mathcal{A}$ in $\mathcal{P}^+(\alpha)$; then, according to Remark 1.3, for g in $\mathrm{GL}_2(\mathbb{R})^+ \setminus \mathbb{R}_+^* \mathrm{SO}(2)$, the flat structures $\mathcal{A}$ and $g\mathcal{A}$ do not induce the same differential structure on the surface S. It was this difficulty that led us to introduce the notion of translation surface in a topological framework.

The action of $\mathrm{GL}_2(\mathbb{R})^+$ that we have just defined is natural. More specifically, it commutes with the group $\mathcal{G}$ of homeomorphisms³ of S. It thus induces an action of $\mathrm{GL}_2(\mathbb{R})^+$ on the space $\tilde{\mathcal{H}}(\alpha)$ which commutes with the action of the mapping class group Γ and an action of $\mathrm{GL}_2(\mathbb{R})^+$ on $\mathcal{H}(\alpha) = \Gamma \backslash \tilde{\mathcal{H}}(\alpha)$.

1.5. **Developing map and the topology of strata.** Let us finish the section by introducing a natural topology on the space $\mathcal{H}(\alpha)$.

Let's fix once and for all a universal covering $\pi : \tilde{S} \to S$. If $\mathcal{A}$ is an element in $\mathcal{P}^+(\alpha)$, there exists a continuous map $D_\mathcal{A} : \tilde{S} \to \mathbb{C}$ such that, for any chart (U, φ, ψ) in $\mathcal{A}$, if V is a connected open set in $\tilde{S}$ such that π induces a homeomorphism from V to U, the function $D_\mathcal{A} - \psi \circ \pi$ is constant on V. The function $D_\mathcal{A}$ is unique up to addition by a constant. By abuse of language, we say that $D_\mathcal{A}$ is the developing map of $\mathcal{A}$. For $g \in \mathrm{GL}_2(\mathbb{R})^+$, the developing map of $g\mathcal{A}$ is $gD_\mathcal{A}$.

We equip the space $\mathcal{P}^+(\alpha)$ with the topology induced by the topology of uniform convergence on compacta for the developing map. More precisely, this is the topology for which a neighbourhood basis of an element $\mathcal{A}_0$ of $\mathcal{P}^+(\alpha)$ consists of the set of $\mathcal{A}$'s in $\mathcal{P}^+(\alpha)$ for which there exists c in $\mathbb{C}$ with

$$\sup_{x \in K} |D_\mathcal{A}(x) - \mathcal{D}_{\mathcal{A}_0}(x) - c| \leq \varepsilon$$

where $\varepsilon > 0$ and a compact subset K of $\tilde{S}$ are fixed. The groups $\mathcal{G}$ and $\mathrm{GL}_2(\mathbb{R})^+$ act then by continuous transformations on $\mathcal{P}^+(\alpha)$. We equip $\mathcal{H}(\alpha)$ and $\tilde{\mathcal{H}}(\alpha)$ with quotient topologies.

One can show that the action of Γ on $\tilde{\mathcal{H}}(\alpha)$ is proper: indeed, the map which associates to a translation structure the underlying complex structure induces a continuous Γ-equivariant map from $\tilde{\mathcal{H}}(\alpha)$ to the Teichmüller space of the surface S and the action of Γ on Teichmüller space is proper ([31], Th. 4.10.5).

²typo. Original: $(U, g\cdot_\alpha, g\psi)$

³french: Original: du groupe $\mathcal{G}$ des homeomorphismes **directs** de S

2. The affine variety structure of strata

In the rest of this exposition, we fix a tuple $(\alpha_1, \dots, \alpha_n)$ of integers ≥ 1 with $\alpha_1 + \cdots + \alpha_n = 2g-2$ and we study the stratum $\mathcal{A}(\alpha)$. We will provide this set with a complex analytic structure that is affine and flat, and invariant under the action of the group $\mathrm{GL}_2(\mathbb{R})^+$. Let's begin by specifying what we mean here by flat affine structure.

2.1. (H, V) **flat structure.** Let M be a topological space, V an affine space (in the sense of elementary affine geometry) and H is a closed subgroup of affine automorphisms of V. A chart from M modelled on V is a tuple (U, φ) where U is an open subset of M and φ is a homeomorphism of U on an open subset of V. Two charts (U_1, φ_1) and (U_2, φ_2) are H-compatible if there exists $h \in H$ for which, for all x in $U_1 \cap U_2$, one has $\varphi_2(x) = h\varphi_1(x)$. A (H, V) atlas on M is a collection $\mathcal{A}$ of charts modelled on V and pairwise compatible, which is maximal for inclusion and such that

$$\bigcup_{(U,\varphi)\in\mathcal{A}} U = M.$$

A flat (H, V) structure on M is the data of such an atlas. It induces on M a structure of a real analytic variety.

In the flat structure that we will now construct for the strata, the space V will have a particular form which we now describe.

Given a finite dimensional real vector space W, with an alternating bilinear form ϖ (which will be defined thanks to a cup-product on cohomology). We put $V = W \oplus W$. Consider an open set V_ϖ consisting of elements (w_1, w_2) of V such that $\varpi(w_1, w_2) > 0$. We denote H the group of linear automorphisms of V which are of the form $(w_1, w_2) \mapsto (gw_1, gw_2)$ where g is a linear automorphism of W that preserves ϖ, so that V_ϖ is stable under H. We say that a flat (H, V)-structure is ϖ-adapted if all of its charts have values in V_ϖ.

The data of such a structure on M defines a germ of the analytic action of $\mathrm{GL}_2(\mathbb{R})^+$ on M, with discrete stabiliser. Indeed, the group $\mathrm{GL}_2(\mathbb{R})$ acts naturally on V and can be identified with $\mathbb{R}^2 \otimes W$. More precisely, for all $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $\mathrm{GL}_2(\mathbb{R})$ and (w_1, w_2) in V, we have

$$g(w_1, w_2) = (aw_1 + bw_2, cw_1 + dw_2).$$

This action commutes with the action of H. It preserves the open set V_ϖ and the stabiliser of an element of V_ϖ is trivial. For $v = (w_1, w_2)$ in V_ϖ, the tangent space at v has its $\mathrm{GL}_2(\mathbb{R})^+$-orbit as vector space $\mathcal{F}_v$ of v generated by the vectors $(w_1, 0)$, $(w_2, 0)$, $(0, w_1)$, $(0, w_2)$.

Given a (H, V)-structure that is ϖ-adapted on M, we can then associate to it the foliation of M by local $\mathrm{GL}_2(\mathbb{R})^+$-orbits, that is to say that the foliation whose tangent bundle $\mathcal{F}$ is such that for any chart (U, φ) of the structure, for any $x \in U$, we have $\mathcal{F}_x = d\varphi(x)^{-1}(X\varphi(x))$. To each element of X of the Lie algebra if $\mathrm{GL}_2(\mathbb{R})^+$ we can associate the vector field $\mathcal{V}^X$ on M such that for the chart (U, φ), for $x \in U$, one has $\mathcal{V}^X_x = d\varphi(x)^{-1}(X\varphi(x))$ (that is, $\mathcal{V}^X$ is the tangent field to the intersection of the chart with the curve $t \mapsto (tX)v$).

Remark 2.1. If the variety M is non-compact, there is no reason for the fields $\mathcal{V}^X$ to define a global $\mathrm{GL}_2(\mathbb{R})^+$ action or one of its covering. This will nevertheless be the case in the flat structures that we are going to build on the strata, since this global $\mathrm{GL}_2(\mathbb{R})^+$-action has been constructed a priori.

Note that, if V is provided with a complex structure J such that $J(w_1, w_2) = (-w_2, w_1)$, a flat (H, V)-structure also induces a complex analytic variety structure.

Suppose that Γ is a discrete group acting properly by diffeomorphisms on M preserving a flat (H, V)-structure. Then, if the action of Γ on M is without fixed point, the variety $\Gamma\backslash M$ is naturally equipped with a flat (H, V)-structure. In the general case, we will call again a flat (H, V)-structure on $\Gamma\backslash M$ the data of a flat (H, V) Γ-invariant structure on M.

2.2. **The Moduli space of Strata.** Recall that we have fixed a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1 + \cdots + \alpha_n = 2g - 2$. We will now seek to build a flat structure of the type described in the previous paragraph on the stratum $\mathcal{H}(\alpha)$. For this we start to build a vector space that will serve us as a model.

Let's choose once and for all an n-tuple $\Sigma = (x_1, \dots, x_n)$ of distinct points in S. We have a long exact sequence of cohomology:

$$0 \to H^0(S, \mathbb{R}) \to H^0(\Sigma, \mathbb{R}) \to H^1(S, \Sigma, \mathbb{R}) \to^p H^1(S, \mathbb{R}) \to 0$$

(where by abuse of notation, we identify Σ with the set $\{x_1, \dots, x_n\}$). In particular, the relative cohomology space $H^1(S, \Sigma, \mathbb{R})$ is of dimension $2g + n - 1$. We will make it play the role of the space W of the preceding paragraph. For that, we will provide an alternating bilinear form. Recall that the space $H^1(S, \Sigma, \mathbb{R})$ is naturally equipped with a cup product, which defines an alternating non-degenerate bilinear form $H^1(S, \mathbb{R}) \times H^1(S, \mathbb{R}) \to H^2(S, \mathbb{R})$[4].

By fixing an orientation of the surface S, we fix a generator of the cyclic group $H^2(S, \mathbb{Z}) \subset H^2(S, \mathbb{R})$. This therefore determines an identification of $H^2(S, \mathbb{R})$ with $\mathbb{R}$ through which we can consider the cup product as an alternating non-degenerate bilinear form

$$\varpi : H^1(S, \mathbb{R}) \times H^1(S, \mathbb{R}) \to \mathbb{R}.$$

By abuse of notation, for x, y in $H^1(S, \Sigma, \mathbb{R})$ we will write $\varpi(x, y)$ for $\varpi(p(x), p(y))$.

Let $W = H^1(S, \Sigma, \mathbb{R})$, $V = W \oplus W = H^1(S, \Sigma, \mathbb{R}^2)$ and let H always be the group of linear automorphisms of V which are of the form $(w_1, w_2) \mapsto (gw_1, gw_2)$ where $g \in \mathrm{GL}(W)$ preserves the bilinear form ϖ. We seek to provide $\tilde{\mathcal{H}}(\alpha)$ and $\mathcal{H}(\alpha)$ with an ϖ-adapted (H, V)-structure in the sense of paragraph 2.1. We will build this structure on a finite cover of $\tilde{\mathcal{H}}(\alpha)$ in which we will only consider structures whose singularities lie in $\{x_1, \dots, x_n\}$. Let's write $\mathcal{P}^+_\Sigma(\alpha) \subset \mathcal{P}^+(\alpha)$ the set of translation structures with conical singularities on S, compatible with the orientation, and whose singularities are $x_1, \dots, x_n$ and have order $\alpha_1, \dots, \alpha_n$. It is stable under the $\mathrm{GL}_2(\mathbb{R})^+$ action. Let $\mathcal{G}_{\alpha,\Sigma}$ the set of homeomorphisms [5] of g of S such that $g\Sigma = \Sigma$ and for all $1 \le i, j \le n$ if $gx_i = x_j$, we have $\alpha_i = \alpha_j$. Let $\mathcal{G}^\circ_{\alpha,\Sigma} \subset \mathcal{G}_{\alpha,\Sigma}$ be the group of homeomorphisms on S which are isotopic to the identity in $\mathcal{G}_{\alpha,\Sigma}$ and $\Gamma_{\alpha,\Sigma} = \mathcal{G}_{\alpha,\Sigma}/\mathcal{G}^\circ_{\alpha,\Sigma}$. Note that the group $\Gamma_{\alpha,\Sigma}$ naturally acts on W while preserving the form ϖ. It therefore acts naturally as a subgroup of H on V.

We put $\mathcal{H}_\Sigma(\alpha) = \mathcal{G}^\circ_{\alpha,\Sigma}\backslash\mathcal{P}^+_\Sigma(\alpha)$. Since $\mathcal{G} = \mathcal{G}^\circ\mathcal{G}_{\alpha,\Sigma}$ (all homeomorphism are isotopic to a homeomorphism that preserves Σ), the set $\Gamma_{\alpha,\Sigma}\backslash\mathcal{H}_\Sigma(\alpha)$ is identified with $\mathcal{H}(\alpha)$. As $\mathcal{G}^\circ_{\alpha,\Sigma}$ is finite index in $\mathcal{G}^\circ \cap \mathcal{G}_{\alpha,\Sigma}$ the set $\tilde{\mathcal{H}}(\alpha)$ is identified with the quotient $\mathcal{H}_\Sigma(\alpha)$ by a finite group.

2.3. **The period map.** Now that we have constructed the space $\mathcal{H}_\Sigma(\alpha)$, we are going to show that it carries a natural flat structure modelled on $V = H^1(S, \Sigma, \mathbb{R}^2)$. For that, we will construct a map called period map $\Pi : \mathcal{H}_\Sigma(\alpha) \to H^1(S, \Sigma, \mathbb{R}^2)$.

Recall that we have fixed a universal covering $\pi : \tilde{S} \to S$. Given an element $\mathcal{A}$, we have associated a developing map $D_{\mathcal{A}} : \tilde{S} \to \mathbb{C}$ defined up to addition by a constant. If

[4]type: Original: $H^2(M, \mathbb{R})$
[5]french: **directs**

$\gamma : [0,1] \to S$ is a continuous path, we choose a continuous covering $\tilde{\gamma} : [0,1] \to \tilde{S}$ of γ. Then, the number

$$c_{\mathcal{A}}(\gamma) = D_{\mathcal{A}}(\tilde{\gamma}(1)) - D_{\mathcal{A}}(\tilde{\gamma}(0))$$

does neither depend on the choice of the developing map $D_{\mathcal{A}}$ nor on the lift of $\tilde{\gamma}$. As, moreover, $c_{\mathcal{A}}(\gamma)$ vanishes when the path γ is constant, the map $c_{\mathcal{A}}$ defines by linearity an element, also denoted $c_{\mathcal{A}}$, of the space $C^1(S, \Sigma, \mathbb{R}^2)$ of cochains of S relative to Σ with coefficients in $\mathbb{C} \simeq \mathbb{R}^2$, still denoted by $c_{\mathcal{A}}$.

It is easy to verify that this cochain is a cocycle.

By construction the map $\mathcal{A} \mapsto c_{\mathcal{A}}$ is equivariant under the action of $\mathrm{GL}_2(\mathbb{R})^+$ (which acts on the cochain coefficients by its natural action on $\mathbb{R}^2$) and under the action of the group $\mathcal{G}_{\alpha,\Sigma}$ (which acts naturally on $C^1(S, \Sigma, \mathbb{R}^2)$). In particular, since $\mathcal{G}^\circ_{\alpha,\Sigma}$ acts trivially on $H^1(S, \Sigma, \mathbb{R}^2)$, the restriction of this map to $\mathcal{P}^+_\Sigma(\alpha)$ defines a map

$$\Pi : \mathcal{H}_\Sigma(\alpha) \to H^1(S, \Sigma, \mathbb{R}^2)$$

which is called the period map of the stratum $\mathcal{H}_\Sigma(\alpha)$. It is $\Gamma_{\alpha,\Sigma}$-equivariant and $\mathrm{GL}_2(\mathbb{R})^+$-equivariant (when $\mathrm{GL}_2(\mathbb{R})^+$ acts on $H^1(S, \Sigma, \mathbb{R}^2) = \mathbb{R}^2 \otimes H^1(S, \Sigma, \mathbb{R})$ through the natural action on $\mathbb{R}^2$ and the trivial action on $H^1(S, \Sigma, \mathbb{R})$.)

The period map will allow us to define an affine structure on the stratum thanks to the remarkable

Theorem 2.2 (Veech [33]). *The period map* $\Pi : \mathcal{H}_\Sigma(\alpha) \to H^1(S, \Sigma, \mathbb{R}^2)$ *is a local homeomorphism.*

So the period map defines on the topological space $\mathcal{H}_\Sigma(\alpha)$ a flat (H, V)-structure which is the inverse image by Π of the tautological structure of $H^1(S, \Sigma, \mathbb{R}^2)$. As Π is $\Gamma_{\alpha,\Sigma}$-equivariant and as $\Gamma_{\alpha,\Sigma}$ acts on V through elements of the group H, this structure descends on $\mathcal{H}(\alpha)$ to a flat affine (H, V)-structure.

It remains for us to check that this structure is $\overline{\omega}$ adapted, that is to say the period map $p \circ \Pi : \mathcal{H}_\Sigma(\alpha) \to H^1(S, \mathbb{R}^2)$ takes its values in the set of pairs (w_1, w_2) in $H^1(S, \mathbb{R}) \oplus H^1(S, \mathbb{R})$ with $\overline{\omega}(w_1, w_2) > 0$. So fix an atlas $\mathcal{A}$ in $\mathcal{P}^+_\Sigma(\alpha)$. Endow S with a real differential structure underlying the flat structure with conical singularities $\mathcal{A}$. The developing map $D_{\mathcal{A}} : \tilde{S} \to \mathbb{C} \simeq \mathbb{R}^2$ is smooth. Write, as it is customary, dx and dy the differential 1-forms associated with the linear coordinates of $\mathbb{R}^2$. The 1-forms $D^*_{\mathcal{A}}dx$ and $D^*_{\mathcal{A}}dy$ can be considered as 1-forms on S.

By construction, $p(\Pi(\mathcal{A}))$ is the image (w_1, w_2) in $H^1(S, \mathbb{R}) \oplus H^1(S, \mathbb{R})$ of the pair of 1-forms $(D^*_{\mathcal{A}}dx, D^*_{\mathcal{A}}dy)$ and so we have

$$\overline{\omega}(w_1, w_2) = \int_S D^*_{\mathcal{A}}(dx \wedge dy). \tag{2}$$

Since $D_{\mathcal{A}}$ is a local diffeomorphism from $\tilde{S} \backslash \pi^{-1}\Sigma$ to $\mathbb{R}^2$ and the orientation of $\mathcal{A}$ is compatible with that of S, this integral is > 0, which was to be demonstrated.

The equivariance of the period map under the group $\mathrm{GL}_2(\mathbb{R})$ implies that it preserves the flat affine structure that we have just constructed on $\mathcal{H}(\alpha)$. More precisely, the orbits of this action are the the maximal leaves of the natural foliations of a $\overline{\omega}$-adapted (H, V)-structure, as introduced in paragraph 2.1.

2.4. **The Hodge bundle.** We end this section by defining a $\mathrm{GL}_2(\mathbb{R})^+$-equivariant bundle over $\mathcal{H}(\alpha)$ which will play a fundamental role in the dynamical study of the $\mathrm{GL}_2(\mathbb{R})^+$-action.

Let's start by temporarily returning to the formalism of paragraph 2.1. Being given a variety M, equipped with a flat (H, V)-structure, we can associate a flat vector bundle with fibre W over M. More precisely, after having fixed a universal covering $\tilde{M} \to M$, the Galois group Λ, the data of a flat (H,V)-structure allows to define a developing map $D : \tilde{M} \to V$ and a holonomy homomorphism $h : \Lambda \to H$ with $D(\lambda x) = h(\lambda)D(x)$, for x in M and λ in Λ. Then, the tangent bundle of M is identified with the quotient $M \times_\Lambda V$ of $\tilde{M} \times V$ by the action of Λ via $\lambda(x,v) = (\lambda x, h(\lambda)v)$.

Suppose that the (H,V) structure is $\overline{\omega}$-adapted where $\overline{\omega}$ is as in paragraph 2.1; in this case, the developing map D has values in the open set $V_{\overline{\omega}}$. Suppose further, to avoid the heavy superfluous notations in the case of strata, that the germ of the action of $\mathrm{GL}_2(\mathbb{R})^+$ is associated with the (H,V)-structure integrates[6] to global action of $\mathrm{GL}_2(\mathbb{R})^+$ on M. In this case, the algebraic structures now provide V both a linear action of Λ on W and on the quotient space $W^{\overline{\omega}} = W/\ker\overline{\omega}$, which allows us to define the flat bundles $M \times_\Lambda W$ and $W^{\overline{\omega}}$. These flat bundles are equipped with a natural $\mathrm{GL}_2(\mathbb{R})^+$-action, which is induced from its action on $\tilde{M} \times W$ (resp. $\tilde{M} \times W^{\overline{\omega}}$) defined by $g(x,w) = (gx, w)$. In particular, the tangent bundle of M can be written as the tensor product $\mathbb{R}^2 \otimes (M \times_\Lambda W)$ of the trivial bundle $M \times \mathbb{R}^2$ with $M \times_\Lambda W$ and it is easy to verify that the tangent action to the action of $\mathrm{GL}_2(\mathbb{R})^+$ is the product tensor action on $\mathbb{R}^2$ and the action described above on $M \times_\Lambda W$.

Being in this abstract context, one can still define $\mathrm{GL}_2(\mathbb{R})^+$-equivariant subbundles of $M \times_\Lambda W$. As above, we will define these subbundles by building subbundles of $\tilde{M} \times W$ that are $(\mathrm{GL}_2(\mathbb{R})^+, \Lambda)$-equivariant. For x in $\tilde{M}$ write $D(x) = (w_1, w_2)$, where w_1 and w_2 are in W and we let $U(x)$ denote the plane of W generated by w_1 and w_2 in W. The restriction of $\overline{\omega}$ to $U(x)$ is non-degenerate. We also write (wrongly, because these bundles are not all flat), $M \times_\Lambda U$, $M \times_\Lambda U_\perp$ and $M \times_\Lambda U_\perp^{\overline{\omega}}$ the bundles obtained from the distributions[7] of the vector subspaces by passing to the quotient under the action of Λ. Then the flat $\mathrm{GL}_2(\mathbb{R})$-equivariant subbundle $M \times_\Lambda U \subset M \times_\Lambda W$ are trivial and the action of $\mathrm{GL}_2(\mathbb{R})$ is identified with its action on $M \times \mathbb{R}^2$. The bundle $M \times_\Lambda U_\perp$ is a $\mathrm{GL}_2(\mathbb{R})$-equivariant complement of $M \times_\Lambda U$.

We will retain this terminology to describe the quotients $\Gamma\backslash M$ where Γ is a discrete group of diffeomorphisms acting without fixed point on M. In particular, we then call the tangent bundle of M the space $\Gamma\backslash(M \times_\Lambda V)$.

Let us return to the case of the strata of Abelian differentials. By definition, the space $W^{\overline{\omega}}$ identifies with the space of cohomology $H^1(S,\mathbb{R})$. We will call this associated bundle on $\mathcal{H}(\alpha)$ the hodge bundle. Write now $\mathcal{H}(\alpha) \times_\Gamma H^1_\perp$ (resp. $\mathcal{H}(\alpha) \times_\Gamma p(H^1_\perp)$) the fibre bundle $M \times_\Lambda U_\perp$ (resp. $M \times_\Lambda U_\perp^{\overline{\omega}}$) as written above.

Remark 2.3. For the case of $g = 1$, we have $H^1_\perp = \{0\}$, so that the tangent action from $\mathrm{GL}_2(\mathbb{R})^+$ identifies with the product action on $\mathcal{H}(\emptyset) \times (\mathbb{R}^2 \otimes \mathbb{R}^2)$. This can be directly seen through elementary arguments of Lie theory and the fact that $\mathcal{H}(\emptyset)$ can be seen as $\mathrm{GL}_2(\mathbb{R})^+/\mathrm{SL}_2(\mathbb{Z})$

3. The Dynamics of the $\mathrm{GL}_2(\mathbb{R})^+$-action

Now that we have introduced the strata of Abelian differentials and described the structures that exists naturally on these spaces, we can state the rigidity theorems for the $\mathrm{GL}_2(\mathbb{R})^+$-action on these strata that have been recently proven by Eskin, Mirzakhani and Mohammadi.

[6] s'integre

[7] french

3.1. **The topological theorem of Eskin-Mirzakhani-Mohammadi.** We keep the notations as previously introduced. Thus we have the stratum $\mathcal{H}(\alpha)$ and we are provided with a (H, V) flat affine structure that is $\overline{\omega}$-adapted.

We will call a linear submanifold of $\mathcal{H}_\Sigma(\alpha)$ to be a connected subvariety $\mathcal{M}$ of $\mathcal{H}_\Sigma(\alpha)$ such that, for every $x \in \mathcal{M}$, there exists an open set U containing x such that the developing map Π realises the diffeomorphism from U to an open set of $V = H^1(S, \Sigma, \mathbb{R}^2)$ and that $\Pi(U \cap \mathcal{M})$ is the intersection of $\Pi(U)$ and a vector subspace V' of V. By connectedness, the subspace V' depends only on $\mathcal{M}$.

We will call a linear submanifold of $\mathcal{H}(\alpha)$ the image of $\mathcal{H}(\alpha)$ of a linear subvariety of $\mathcal{H}_\Sigma(\alpha)$. Let $B \subset \mathrm{GL}_2(\mathbb{R})^+$ the subgroup of upper triangular matrices.

Theorem 3.1 (Eskin-Mirzakhani-Mohammadi [12])**.** *Let x be in $\mathcal{H}(\alpha)$. Then the closure $\overline{Bx}$ of the orbit of x under B is the orbit closure $\overline{\mathrm{GL}_2(\mathbb{R})^+x}$ of x under $\mathrm{GL}_2(\mathbb{R})^+$ and this set is a linear submanifold of $\mathcal{H}(\alpha)$.*

When $g = 2$ (and then $\alpha = (1, 1)$ or $\alpha = (2)$) this rigidity property is due to McMullen [26], which also gives a precise classification.

3.2. **The metric theorem of Eskin-Mirzakhani.** Theorem 3.1 is a consequence of a classification result of invariant measures. This follows from the technique that has become classical since Ratner's work [29],[30] in homogeneous dynamics, a technique that will be explained later.

Let's state the result now. For this we need to introduce measures associated to linear subvarieties. Let $\mathcal{M} \subset \mathcal{H}_\Sigma(\alpha)$ be a linear submanifold and let $V' \subset V$ be the associated vector subspace. Let's choose a Lebesgue measure ν' on V'. There exists then a unique Radon measure $\nu_\mathcal{M}$ on $\mathcal{H}_\Sigma(\alpha)$, where the restriction of Π induces a diffeomorphism to an open subset of V such that the measure $\Pi_*(\nu_\mathcal{M}|_U)$ is equal to the restriction of ν' to $\Pi(U)$. The measure $\nu_\mathcal{M}$ is determined by $\mathcal{M}$ up to multiplication with a scalar > 0.

Let $k \geq 0$ be an integer and ν a Radon measure on $\mathcal{H}(\alpha)$. We can associate with ν in a natural way a $\Gamma_{\alpha,\Sigma}$-invariant Radon measure $\tilde{\nu}$ on $\mathcal{H}_\Sigma(\alpha)$. We'll say that ν is linear (of dimension k) if, for every x in the support of $\tilde{\nu}$, there exists a linear subvariety $\mathcal{M}$ of dimension k of $\mathcal{H}_\Sigma(\alpha)$ and an open set U of $\mathcal{H}_\Sigma(\alpha)$ containing x such that $\tilde{\nu}|_U = \nu_\mathcal{M}|_U$.

Remark 3.2. When $g = 1$, the action of $\mathrm{GL}_2(\mathbb{R})^+$ on $\mathcal{H}(\emptyset)$ is identified with its action on $\mathrm{GL}_2(\mathbb{R})^+/\mathrm{SL}_2(\mathbb{Z})$. This does not preserve a finite Borel measure. If one wants to study the ergodic properties of this action, it is therefore necessary to restrict oneself to the action of the group $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$.

In the general case, we will now describe a function $\mathcal{H}(\alpha) \to \mathbb{R}_+^*$ whose level lines will be $\mathrm{SL}_2(\mathbb{R})$-invariant. For $\mathcal{A}$ in $\mathcal{P}^+(\alpha)$, let us define the area of the translation surface $(S, \mathcal{A})$ as the number appearing in ((2)), that is, the area of a fundamental domain of S under the inverse image of the developing map $D_\mathcal{A}$ of the standard volume form on $\mathbb{R}^2$. This area function is $\mathcal{G}$-invariant and therefore factors to a function $a : \mathcal{H}(\alpha) \to \mathbb{R}_+^*$. One can also say that, if x is an element in $\mathcal{H}_\Sigma(\alpha)$ and if $\Pi(x)$ is of the form (w_1, w_2) with w_1, w_2 in $W = H^1(S, \Sigma, \mathbb{R})$, then $a(x) = \overline{\omega}(w_1, w_2)$. For g in $\mathrm{GL}_2(\mathbb{R})^+$ and x in $\mathcal{H}(\alpha)$, we have $a(gx) = (\det g)a(x)$.

We define $\mathcal{H}_1(\alpha)$ as the set of x in $\mathcal{H}(\alpha)$ with $a(x) = 1$. The map that identifies a point x in $\mathcal{H}(\alpha)$ associates the tuple $(\sqrt{a(x)}, \frac{1}{\sqrt{a(x)}}x)$ gives an identification of $\mathcal{H}(\alpha)$ and $\mathbb{R}_+^* \times \mathcal{H}_1(\alpha)$.

Let $k \geq 0$ be an integer and ν be a Radon measure on $\mathcal{H}_1(\alpha)$. We say that ν is affine (of dimension k) if the measure $t^k dt \otimes \nu$ on $\mathcal{H}(\alpha) \simeq \mathbb{R}_+^* \times \mathcal{H}_1(\alpha)$ is linear of dimension $k+1$. We will also say that a subset $\mathcal{M}$ of $\mathcal{H}_1(\alpha)$ is an affine subvariety of the set $\mathbb{R}_+^* \times \mathcal{H}_1(\alpha) \subset \mathcal{H}(\alpha)$ is a linear subvariety. The support of an affine measure is an affine subvariety.

Remark 3.3. The terms affine measure and affine subvariety do not seem very appropriate. Their use, however, seems to be the consensus of the authors[8].

We denote $P = B \cap \mathrm{SL}_2(\mathbb{R})$ the upper triangle group in $\mathrm{SL}_2(\mathbb{R})$.

Theorem 3.4 (Eskin-Mirzakhani [11])**.** *Let ν be a P-invariant ergodic probability measure on $\mathcal{H}_1(\alpha)$. Then ν is $\mathrm{SL}_2(\mathbb{R})$-invariant and affine.*

In genus 2, the classification of $\mathrm{SL}_2(\mathbb{R})$-invariant measures is also due to McMullen [26].

3.3. **Another case of rigidity: The theorems of Ratner.** [9]

Theorems 3.1 and 3.4 are answers [10] to conjectures that have been formulated several years ago by specialists in the subject by analogy with the famous theorems of Ratner. We recall here briefly the statements of these theorems.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Given an element X of $\mathfrak{g}$, we say that X is ad-nilpotent if the endomorphism ad_X of $\mathfrak{g}$ is nilpotent. We also say that the group parametrised by $t \mapsto \exp(tX)$ is Ad-unipotent.

Theorem 3.5 (Ratner [29])**.** *Let Γ be a discrete subgroup of G, and H a closed subgroup of G generated by a one-parameter* Ad*-unipotents and ν a Borel probability measure that is H-invariant and ergodic on G/Γ. Then there exists a subgroup L of G and a point x of X*[11] *such that $\nu(Lx) = 1$ and ν is L-invariant.*

Note that, if H is the image under a homomorphism $\mathrm{SL}_2(\mathbb{R}) \to G$, it is as well generated by Ad-unipotent elements.

By the procedure already mentioned above, we deduce from this metric theorem a topological statement:

Theorem 3.6 (Ratner [30])**.** *Suppose that Γ is a lattice in G and that H is as above. If x is a point in G/Γ, there exists a closed subgroup L of G such that $\overline{Hx} = Lx$.*

In view of these statements, it would be legitimate to ask whether in Theorem 3.1 and 3.4 the group P can be replaced by the group N of matrices of the form $\left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$, $t \in \mathbb{R}$.

In this case, counterexamples to this extension were constructed by Smillie and Weiss (private communication).

3.4. **Equidistribution.** We will now sketch the proof of Theorem 3.1 from Theorem 3.4. As we said above, we will follow a relatively classical argument, which consists of establishing from the classification theorem 3.4 to a result about equidistribution.

In Ratner's theorems, the equidistribution concerns orbits of a one parameter Ad-unipotent group and it is based on an analytical technique on behaviour of these flows close to invariant subvarieties and close to infinity of G/Γ, a technique developed by Dani and Margulis [6] which concerns the behaviour at infinity.

[8]french. Original: Leur emploi semble neanmoins faire consensus chez les auteurs.

[9]Changed "theory" to "theorems" as experts point out that the use of "theory" undermines contributions by others.

[10]french

[11]typo

In this case, we will establish the equidistribution property for any $\mathrm{SL}_2(\mathbb{R})$-orbit. For this the authors use a method introduced by Eskin, Margulis and Mozes [9], inspired by probabilistic arguments, which we will now describe.

Consider a locally compact topological space (separable, σ-compact) X, equipped with a continuous group action of $\mathrm{SL}_2(\mathbb{R})$. For t, θ in $\mathbb{R}$, we write

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \text{ and } r_\theta = \begin{pmatrix} \cos\theta & -\sin(\theta) \\ \sin\theta & \cos(\theta) \end{pmatrix}.$$

If $f : X \to \mathbb{R}_+$ is a continuous function, we put, for $x \in X$,

$$A_t f(x) = \int_0^{2\pi} \int_0^{2\pi} f(r_{\theta_1} a_t r_{\theta_2} x) d\theta_1 d\theta_2 = \int_{\mathrm{SL}_2(\mathbb{R})} f(gx) d\mu_t(g)$$

where μ_t is the left and right $\mathrm{SO}(2)$-invariant probability measure (sometimes said bi-$\mathrm{SO}(2)$-invariant) on the set $\mathrm{SO}(2)a_t\mathrm{SO}(2)$.

Remark 3.7. The quotient space $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ identifies with the hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ through the map $g \mapsto gi$. The set $\mathrm{SO}(2)a_t\mathrm{SO}(2)$ is the inverse image under the hyperbolic circle centred at i and radius $2t$ under this map.

We will give a criterion of recurrence for the action of $\mathrm{SL}_2(\mathbb{R})$ on X for which we consider the operators A_t as Markov operators and we demand that they satisfy classical recurrence properties from probability theory (see [27],[28]). Then we will say that the action of $\mathrm{SL}_2(\mathbb{R})$ in X is exponentially recurrent of there exists a continuous proper $\mathrm{SO}(2)$-invariant function $f : X \to \mathbb{R}_+$ and $C > 0$ such that, for all $0 < c \leq 1$, there exists $t_0 > 0$ with, for $t \leq t_0$ and x in X,

$$A_t f(x) \leq c f(x) + C. \tag{3}$$

Proposition 3.8 (Eskin-Mirzakhani-Mohammadi [12])**.** *Let $\mathcal{M}$ be a $\mathrm{SL}_2(\mathbb{R})$-invariant affine subvariety of $\mathcal{H}(\alpha)$. Then the space $\mathcal{H}(\alpha) \setminus \mathcal{M}$ is exponentially recurrent for the action of* $\mathrm{SL}_2(\mathbb{R})$.

In the case where $\mathcal{M}$ is empty, the fact that the $\mathrm{SL}_2(\mathbb{R})$-action on $\mathcal{H}(\alpha)$ is exponentially recurrent is due to Athreya [1].

This recurrence property being established, an ingredient we still lack to deduce Theorem 3.1 from Theorem 3.4 is the following result of countability, which we will say that an orbit does not have much choice of affine subvarities to accumulate to.[12] This result is also an analogue of a phenomenon that appears in Ratner's theory [30].

Proposition 3.9 (Eskin-Mirzakhani-Mohammadi [12])**.** *The set of measures of affine* $\mathrm{SL}_2(\mathbb{R})$-*invariant probability measures on $\mathcal{H}(\alpha)$ is countable.*

We can then establish the

Proof of Theorem 3.1. Let $\mathfrak{M}$ be the set of affine $\mathrm{SL}_2(\mathbb{R})$-invariant and ergodic measures on $\mathcal{H}(\alpha)$. Fix x in $\mathcal{H}(\alpha)$.

Let's start by noticing that there is an element ν_x in $\mathfrak{M}$ whose support $\mathcal{M}_x$ contains x and is minimal with that property. Indeed, let's choose ν_x so that the support $\mathcal{M}_x$ is of minimal dimension. If ν is an element in $\mathfrak{M}$, whose support $\mathcal{M}$ contains x, has the same dimension as $\mathcal{M}_x$ and is included in $\mathcal{M}_x$, as $\mathcal{M}$ is open in $\mathcal{M}_x$, then we have $\nu_x|_{\mathcal{M}} = \nu$ and thus, by ergodicity, $\nu = \nu_x$, which had to be demonstrated.

[12]french: Original: qui nous dira qu'une orbite n'a pas beaucoup de choix de sous-varietes affines ou s'accumuler.

Let us now show that $\mathrm{SL}_2(\mathbb{R})x = \mathcal{M}_x$. Let ν_∞ be an accumulation point when $p \to \infty$ of the sequence of probability measures

$$\nu_p = \frac{1}{p}\sum_{k=0}^{p-1} \mu_1^{*k} * \delta_x$$

on $\mathcal{H}(\alpha)$ for vague convergence of measures. We will show that $\nu_\infty = \nu_x$ which will complete the proof.

Let's start by showing that ν is a probability measure, that is, the mass of ν_p remains concentrated on compact parts of $\mathcal{H}(\alpha)$. Let's apply Proposition 3.8 with $\mathcal{M} = \emptyset$. We then have a proper continuous function $f : \mathcal{H}(\alpha) \to \mathbb{R}_+$ and $C > 0$ that satisfy (3). We will estimate

$$A_1^k f(x) = \int_{\mathrm{SL}_2(\mathbb{R})} f(gx) d\mu_1^{*k}(g)$$

for large k. For this note that, since the measure μ_1^{*k} is bi-SO(2)-invariant, it can be written as

$$\mu_1^{*k} = \int_{\mathbb{R}_+} \mu_t d\rho_k(t)$$

where ρ_k is a probability measure on $\mathbb{R}_+$. Let $t_0 > 0$ be such that (3) is valid, with $c = 1$. By a theorem of Furstenberg on random matrix products [15],[5], for all $t_0 > 0$, we have

$$\rho_k([0, t_0]) \to_{k\to\infty} 0 \tag{4}$$

(the norm of a random product of k matrices tends to infinity with k). Let $\varepsilon > 0$ and $K = f^{-1}([0, \frac{1}{\varepsilon}]) \subset \mathcal{H}(\alpha)$, so that K is compact by properness of f. For all $t \geq t_0$, one has according to Chebyshev's inequality,

$$\mu_t * \delta_x(K^c) \leq \varepsilon A_t f(x) \leq \varepsilon(f(x) + C)$$

and thus

$$\mu_1^{*k}(K^c) \leq \rho_k([0, t_0]) + \varepsilon(f(x) + C).$$

According to (4), we have

$$\limsup_{p\to\infty} \nu_p(K^c) \leq \varepsilon(f(x) + C).$$

Since this is true for all $\varepsilon > 0$, ν_∞ is a probability measure.

Remark 3.10. The use of $c < 1$ in the definition of exponential recurrence would allow to show that we actually have

$$\limsup_{p\to\infty} \nu_p(K^c) \leq \varepsilon C.$$

Now let us be given ν in $\mathfrak{M}$ whose support $\mathcal{M}$ is properly included in $\mathcal{M}_x$. By minimality, one has $x \notin \mathcal{M}$. By the above reasoning, applying now Proposition 3.8 to $\mathcal{M}$, we show that $\nu_\infty(\mathcal{M}) = 0$.

We will now show that ν_∞ is $\mathrm{SL}_2(\mathbb{R})$-invariant. Let's start by remarking that, by construction, ν_∞ is A_1^*-invariant (where A_1^* is the adjoint operator of A_1, which acts on Borel measures). By another theorem of Furstenberg [16],[3], such a measure is written as an average of probability measures

$$\nu_\infty = \int_{\mathrm{SL}_2(\mathbb{R})/P} \nu_{\infty,\xi} d\sigma(\xi),$$

where σ is a SO(2)-invariant probability measure on $\mathrm{SL}_2(\mathbb{R})/P \simeq \mathbb{P}^1_{\mathbb{R}}$ and where, for all $\xi = gP$ in $\mathrm{SL}_2(\mathbb{R})/P$, the measure $\nu_{\infty,\xi}$ is gPg^{-1}-invariant. By Theorem 3.4, the measure $\nu_{\infty,\xi}$ is $\mathrm{SL}_2(\mathbb{R})$-invariant, and thus ν_∞ is $\mathrm{SL}_2(\mathbb{R})$-invariant.

The measure ν_∞ decomposes into an average of $\mathrm{SL}_2(\mathbb{R})$-invariant and ergodic measures. According to Theorem 3.4, these ergodic measures are still elements of $\mathfrak{M}$. Since, according to Proposition 3.9, the set $\mathfrak{M}$ is countable, one can write

$$\nu_\infty = \sum_{\nu \in \mathfrak{M}; \operatorname{supp} \nu \subset \mathcal{M}_x} a_\nu \nu.$$

Now, for all ν in $\mathfrak{M}$, if the support $\mathcal{M}$ of ν is properly included in $\mathcal{M}_x$, one has

$$a_\nu \leq \nu_\infty(\mathcal{M}) = 0.$$

It follows $\nu_\infty = \nu_x$ which had to be demonstrated. □

3.5. **Applications to billiards.** Theorem 3.1 implies consequences for counting periodic trajectories in billiards with rational angles. Indeed, following a classical procedure (see for example [34]), to any such billiard, one can associate a translation surface. The dynamical properties of the billiard can be reinterpreted for the translation flow on the surface. If the billiard is rectangular, this translation flow is a torus and elementary counting arguments allow to show that the number of (cylinders of) periodic trajectories of length $\leq T$ is equal to a multiple to T^2. In the general case, we have the

Theorem 3.11 (Masur ([24],[25]))**.** *Given a rational polygonal billiard, there exists constants $0 < c_1 \leq c_2$ such that the number $N(T)$ of cylinders of periodic trajectories of length $\leq T$ satisfies*

$$c_1 T^2 \leq N(T) \leq c_2 T^2.$$

Theorems 3.1 and3.4 imply, following a method that was known [10],

Theorem 3.12 (Eskin-Mirzakhani-Mohammadi [12])**.** *Given a rational billiard, there exists a constant $c > 0$ such that the number $N(T)$ of cylinders of periodic trajectories of length $\leq T$ satisfies*

$$e^{-T} \int_1^T \frac{N(S)}{S^2} \frac{dS}{S} \to_{T \to \infty} c.$$

The question of whether $N(T) \sim_{T\to\infty} cT^2$ is open.

4. Linear Cocycles

Now that we have seen that Theorem 3.1 derives from Theorem 3.4 by a relatively standard procedure - though not necessarily easy to implement - we will try to give elements of the proof of the remarkable Theorem 3.4.

As we said above, Theorem 3.1 is intended as an analogue for the action of $\mathrm{SL}_2(\mathbb{R})$ on the strata of Abelian differentials from Ratner's theorem 3.5, its proof borrows some of the methods developed for the study of homogeneous dynamics [4],[23].

The general strategy of this type of theorem is as follows. We have a group H acting by diffeomorphisms on a variety X equipped with a probability measure ν. It is sought to show that ν must satisfy certain geometrical conditions (being affine, being homogeneous under the action of a subgroup). For this we apply abstract theorems of ergodic theory which guarantee that for ν-almost every point x, some sequence of subsets $H_{n,x}$ of the orbits Hx of x equidistribute in X. We then apply these theorems to two points x and y very close in X, in order to find h and n in H such that on the one hand $hx \in H_{n,x}$ and $hy \in H_{n,y}$

and, on the other hand, the distance between hx and hy is of prescribed size (and no longer arbitrarily small). The equidistribution of $H_{n,x}$ and $H_{n,y}$ must force hx and hy to be in sets of big measure where certain measurable H-equivariant functions become continuous when restricted to these sets (such sets exists by Lusin's theorem). The fact that x and y are close and that hx and hy are at a controlled distance makes it possible to show that certain objects associated with ν and the geometry of space have properties of invariance by transformations that make it possible to go from hx to hy.

I will try to make this last paragraph clearer in the rest of the text.

Nevertheless, the reader may have understood that an important problem of the subject consists in controlling how fast, from two points x and y close together, the orbits Hx and Hy move away from each other. For this we analyse the tangent action of the group on the tangent bundle TX.

It is here that there is a very important difference between the case of homogeneous spaces and that of strata of Abelian differentials.

Indeed, if G is a Lie group and Γ is a discrete subgroup of G, the tangent action of G on G/Γ is constant. More precisely, if $\mathfrak{g}$ is the Lie algebra of G, by identifying $\mathfrak{g}$ and the Lie algebra of right-invariant vector fields of G, we obtain an isomorphism of TG with $G \times \mathfrak{g}$, through which the action to the right of an element g of G reads

$$(h, X) \mapsto (hg, X)$$

and the action on the left becomes the transformation

$$(h, X) \mapsto (gx, \mathrm{Ad}(g)X)$$

where Ad denotes the adjoint action of G on $\mathfrak{g}$. This structure being invariant on the right, factors through G/Γ and gives an isomorphism between TG/Γ and $G/\Gamma \times \mathfrak{g}$, through which the tangent action to the action of g always reads

$$(x, X) \mapsto (gx, \mathrm{Ad}(g)X).$$

In particular, the asymptotic behaviour of this tangent action along an orbit $g^n x$, $n \geq 0$ does not depend on x and is determined by the Jordan decomposition of the linear endomorphism $\mathrm{Ad}(g)$.

In the strata of Abelian differentials, the tangent bundle does not have an $\mathrm{SL}_2(\mathbb{R})$-equivariant trivialisation. In contrast, the flat affine structure is $\mathrm{SL}_2(\mathbb{R})$-invariant, so that the tangent action is somehow locally constant. More precisely, if $(g_t)_{0 \leq t \leq 1}$ is a continuous curve in $\mathrm{SL}_2(\mathbb{R})$ with $g_0 = e$, and if x and y are two points close to $\mathcal{H}(\alpha)$ such that, for all $0 \leq t \leq 1$, $g_t x$ and $g_t y$ remain close, one can naturally identify the tangent spaces to $\mathcal{H}(\alpha)$ at x and y (resp. $g_1 x$ and $g_1 y$) so that the differential of the action of g_1 is given by the same linear map in x and y.

The use of this idea plays a fundamental role in the work of Eskin and Mirzakhani. It is formulated with the theory of Osseledets, that is to say that the general study of asymptotic behaviour of the tangent action over an invariant measure for the action of a group on a differential variety. We will start with recalling the basics of this theory.

4.1. **Cocycles over an ergodic action.** We recall here the usual language of cocyles in ergodic theory.

Let G be a locally compact topological group (separable and countable unions of compacta) acting measurably on a probability Lebesgue space (X, ν) while preserving the measure ν (recall that a space is Lebesgue if it is a probability space isomorphic to the disjoint union of an interval, equipped with the Lebesgue measure and a number of at most countable countably many atoms). We assume that ν is G-ergodic. Let H be another locally compact topological group (separable and countable union of compacta).

A cocycle with values in H over the action of G on X is a measurable map

$$\sigma : G \times X \to H$$

such that for all g_1, g_2 in G, for ν-almost every x in X, one has

$$\sigma(g_1 g_2, x) = \sigma(g_1, g_2 x)\sigma(g_2, x). \tag{5}$$

Remark 4.1. In the following, we will eventually allow ourselves to reverse the quantifiers in formula (5) and similar formulas. This is possible thanks to general theorems about the action of locally compact separable groups that are union of compacta on Lebesgue spaces (see for example [35]).

Example 4.2. Suppose that X is a differential manifold of dimension d on which G acts by diffeomorphisms. Then the tangent bundle of X is measurably trivial: there is a measurable isomorphism from the fibres of TX to $X \times \mathbb{R}^d$. The tangent action of G on TX is then defined by a measurable mapping $\sigma : G \times X \to \mathrm{GL}_d(\mathbb{R})$. The formula of the derivative of these maps results in the fact that the map σ is a cocycle. This argument extends to any fibred action of G on a vector bundle above X.

Let Y be a set with an action of H. Then to a cocycle $\sigma : G \times X \to H$, we can associate the skew action on the set $X \times Y$ defined by $g(x, y) = (gx, \sigma(g, x)y)$, $g \in G$, $x \in X$, $y \in Y$. The relation (5) ensures precisely that this formula defines an action.

Two cocycles $\sigma, \sigma' : G \times X \to H$ are cohomologous if there exists a map $\varphi : X \to H$ with, for all g in G and ν-almost all x in X

$$\sigma'(g, x) = \varphi(gx)\sigma(g, x)\varphi(x)^{-1}.$$

In this case, the skew actions associated to σ and σ' are conjugated.

4.2. **The Zariski Closure.** We will associate to a cocycle with values in an algebraic group H a conjugacy class of algebraic subgroups of H. By algebraic group, we will always mean the group of real points of an affine algebraic group defined over $\mathbb{R}$, that is, in elementary language, a subgroup of a matrix group, defined by polynomial equations in the coordinates.

Let G and (X, ν) as before and H a locally compact, separable and σ-compact. Let $\sigma : G \times X \to H$ be a cocycle and L be a closed subgroup of H[13]. If σ is cohomologous to a cocycle with values in L, there exists a map $\varphi : X \to H/L$ such that, for every g in G and ν-almost all x in X, $\varphi(gx) = \sigma(g, x)\varphi(x)$, that is to say that the fibration $X \times H/L \to X$ possesses a G-equivariant section for the skew action. Conversely, if such a map φ exists, the existence of a Borel section for the quotient map $H \to H/L$ ensures that σ is cohomologous to a cocycle with coefficients in L.

Proposition 4.3 (Zimmer [35]). *Let H be a real algebraic group and $\sigma : G \times X \to H$ be a cocycle. There is an algebraic subgroup L of H such that σ is cohomologous to a cocycle with coefficients in L, and if L' is another algebraic subgroup with this property, L' contains a conjugate of L (that is to say that one has $hLh^{-1} \subset L'$ for some h in H).*

We say that the group L is the Zariski-Zimmer closure of the cocycle σ (this is abuse of language, since L is only defined up to conjugation).

[13] typo

Proof. This follows from the following remark: if L_1 and L_2 are algebraic subgroups of L such that σ is cohomologous on the one side to a cocycle with coefficients in L_1 and on the other side cohomologous to a cocycle with coefficients in L_2, then σ is cohomologous to a cocycle with coefficients in a subgroup of the form $L_1 \cap hL_2h^{-1}$ for some element h of H.

Indeed, for $i = 1, 2$, there exists a map φ_i from X to H/L_i such that, for every g in G, for ν-almost all x in X, we have $\varphi_i(gx) = \sigma(g,x)\varphi_i(x)$. We put $\varphi = (\varphi_1, \varphi_2)$ and we equip $Y = H/L_1 \times H/L_2$ with the product action of H. We still have $\varphi(gx) = \sigma(g,x)\varphi(x)$. Now since Y is a quasi-projective variety, equipped with an algebraic action of H, the quotient $H\backslash Y$ has a countable basis. By the ergodicity of the action from G on X, the map φ thus takes its values in a single H-orbit in Y. This is of the form H/L where L is the stabiliser of a point of Y. These stabilisers are of the form $h_1L_1h_1^{-1} \cap h_2L_2h_2^{-1}$. □

4.3. **Actions of amenable groups.** We are now going to use the concept of the Zariski-Zimmer closure and the theory of the linear groups to show that, if G is amenable, a cocycle with values in an algebraic group $H \subset \mathrm{GL}_d(\mathbb{R})$ is essentially cohomologous to a cocycle with values in a matrix group that is triangulated by blocks, where the blocks are conformal. It is this result, applied to the action of the subgroup $P \subset \mathrm{SL}_2(\mathbb{R})$ in $\mathcal{H}(\alpha)$, which will allow us to replace the arguments based on Jordan reduction of usual linear endomorphisms as used in the study of dynamical systems on homogeneous spaces.

The assumption that G is amenable will be used in the form of the following lemma, which is a fibred version of the fact that any continuous action of G on a compact space preserves a probability measure.

Lemma 4.4. *Let $\sigma : G \times X \to H$ be a cocycle and Y be a compact metric space, with a continuous action of H. Suppose that G is amenable. Then the skew action of G on $X \times Y$ has an invariant measure whose projection on X is ν.*

Proof. We will need to build a measurable map $x \mapsto \rho_x$ from X to the space of Borel probability measures on Y such that, for all g in G, for ν-almost all x in X, we have $\rho_{gx} = \sigma(g,x)_*\rho_x$. We will show that such map exists by realizing that it as a fixed point for some continuous action of G on a convex compact part of a locally convex topological vector space. For this consider the Banach space

$$E = L^1(X, \nu, \mathcal{C}^0(Y))$$

of essential[14] classes of functions $x \mapsto f_x$ from X to $\mathcal{C}^0(Y)$ which are measurable (for the Borel structure of the separable Banach space $\mathcal{C}^0(Y)$) and such $\int_X \|f_x\|_\infty d\nu(x)$. Identify the dual space of $\mathcal{C}^0(Y)$ with the space $\mathcal{M}(Y)$ of complex Borel measures on Y and write $\mathcal{P}(Y) \subset \mathcal{M}(Y)$ for the space of probability measures. Since (X, ν) is a Lebesgue space and $\mathcal{C}^0(Y)$ is separable, the dual space E' of E naturally identifies with the space of essential classes of maps $x \mapsto \rho_x$ of X to $\mathcal{M}(Y)$ which are measurable (for the Borel structure induced by the weak* topology of $\mathcal{M}(Y)$) and are essentially bounded. Let's write $E'_1 \subset E'$ be the set consisting of maps $x \mapsto \rho_x$ which essentially take their values in $\mathcal{P}(Y)$. Then, according to the Banach-Alaoglu theorem, E'_1 is a convex compact subset of E' for the weak* topology. As the skew action of G on $X \times Y$ induces a weak* continuous action on E' which preserves E'_1, G fixes a point in E'_1 which was to be proven. □

In our constructions, the space Y will be a certain projective space $\mathbb{P}^{d-1}_{\mathbb{R}}$. We will then ask to apply to the constructed measures a classical lemma on the structure of the stabiliser

[14]french

of a measure on $\mathbb{P}^{d-1}_{\mathbb{R}}$ in $\mathrm{GL}_d(\mathbb{R})$. Recall that a subgroup L of $\mathrm{GL}_d(\mathbb{R})$ acts irreducibly on $\mathbb{R}^d$ if and only if $\{0\}$ and $\mathbb{R}^d$ are the only L-invariant subspaces of $\mathbb{R}^d$[15]

Lemma 4.5 (Furstenberg)**.** *Let L be a subgroup of $\mathrm{GL}_d(\mathbb{R})$ that acts irreducibly on $\mathbb{R}^d$. Suppose that L preserves a probability measure on $\mathbb{P}^{d-1}_{\mathbb{R}}$. Then there exists a finite index subgroup M of L and a decomposition of $\mathbb{R}^d$ as direct sum $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_k$ such that, for $1 \leq i \leq k$, M preserves V_i and the image of M in $\mathrm{PGL}(V_i)$ is compact.*

Proof. Let's start by showing that, if L is a subgroup of $\mathrm{GL}_d(\mathbb{R})$, which preserves a probability measure ρ on $\mathbb{P}^{d-1}_{\mathbb{R}}$, and if the image of L in $\mathrm{PGL}_d(\mathbb{R})$ is not compact, the measure ρ is concentrated on the union of two projective subspaces of $\mathbb{P}^{d-1}_{\mathbb{R}}$. Indeed, by assumption, L contains a sequence of elements g_n whose decomposition in Cartan form is

$$k_n \begin{pmatrix} a_{1,n} & & 0 \\ & \ddots & \\ 0 & & a_{d,n} \end{pmatrix} \ell_n$$

(with k_n, ℓ_n in $O(d)$ and $a_{1,n} \geq \cdots \geq a_{d,n} > 0$) and which tend to infinity in $\mathrm{PGL}_d(\mathbb{R})$. After extracting a subsequence, we can assume that for some k, ℓ in $O(d)$ and $1 \leq i \leq d-1$, one has $k_n \to k$, $\ell_n \to \ell$ and

$$\frac{a_{i,n}}{a_{i+1,n}} \to \infty.$$

Then ρ is necessarily concentrated on the whole set

$$k\mathbb{P}(\mathbb{R}^i \times \{0\}) \bigcup \ell^{-1}\mathbb{P}(\{0\} \times \mathbb{R}^{d-i}).$$

Suppose that L acts irreducibly on $\mathbb{R}^d$ and let $\mathcal{V}$ be the set of non-zero subspaces of $\mathbb{R}^d$ such that $\rho(\mathbb{P}(V)) > 0$ and that the dimension of V is minimal with this property. For all $V_1 \neq V_2$ in $\mathcal{V}$, we have

$$\rho(\mathbb{P}(V_1) \bigcup \mathbb{P}(V_2)) = \rho(\mathbb{P}(V_1)) + \rho(\mathbb{P}(V_2))$$

and therefore the set $\mathcal{W}$ of elements of $\mathcal{V}$ whose image is in $\mathbb{P}^{d-1}_{\mathbb{R}}$ is of maximal measure is finite. Since L preserves $\mathcal{W}$, the elements of $\mathcal{W}$ generate $\mathbb{R}^d$. For W in $\mathcal{W}$, the stabiliser of W in L preserves the restriction of ρ in $\mathbb{P}(W)$ and this measure does not charge a projective subspace of $\mathbb{P}(W)$. This group therefore as compact image in $\mathrm{PGL}(W)$. The lemma follows. □

To complete the general reduction of the cocycles above for amenable group actions, we will still need a tool to circumvent the problem that arises when the finite index subgroup M of Lemma 4.5 is a proper subgroup of L. We will call a finite extension of the action of G on X an action of G on a space of the form $\tilde{X} = X \times F$ which is the skew action defined by a cocycle with values in the group of permutations of the set F. In other words, it is an action on $X \times F$ such that the projection on X is G-equivariant. We provide X with the product measure of ν and the normalised counting measure on F. Any cocycle above X can then be seen as a cocycle above X without being a cocycle above $\tilde{X}$. On the other hand, two cocycles can be cohomologous above $\tilde{X}$ without being so above X.

Let L be a subgroup of $\mathrm{GL}_d(\mathbb{R})$ and $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k = \mathbb{R}^d$ a flag of $\mathbb{R}^d$. We say that the group L is triangular with conformal blocks with respect to the flag (V_i) if for all $1 \leq i \leq k$, one has $LV_i = V_i$ and the image of L in $\mathrm{PGL}(V_i)$ is compact. In other words,

[15] typo

there exists a basis adapted to the flag (V_i) in which the matrices of the elements of L are of the form

$$\begin{pmatrix} e^{\lambda_1}U_1 & * & * & * \\ 0 & e^{\lambda_2}U_2 & * & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_k}U_k \end{pmatrix} \tag{6}$$

with $\lambda_i \in \mathbb{R}$, $U_i \in O(d_i)$, $d_i = \dim V_i - \dim V_{i-1}$, $1 \le i \le k$.

From the two preceding lemmata, we deduce the

Proposition 4.6. *Suppose that the group G is amenable. Let $\sigma : G \times X \to \mathrm{GL}_d(\mathbb{R})$ be a cocycle. There exists an action of G on a finite extension $\tilde{X}$ such that the Zariski-Zimmer closure of σ, seen as a cocycle over $\tilde{X}$, is triangular with conformal blocks. In other words, over $\tilde{X}$, the cocycle σ is cohomologous to a cocycle with values in the group of matrices of the form* (6).

Proof. Let L be the Zariski-Zimmer closure of σ. We can suppose that σ takes values in L. Let W be an irreducible subquotient of the action of L on $\mathbb{R}^d$, that is to say $W = U/V$. where $U \supset V$ are L-invariant subspaces of $\mathbb{R}^d$ and that the action of L on W[16] is irreducible. According to Lemma 4.4 there is a measurable map $x \mapsto \rho_x$ from X to the space $\mathcal{P}(\mathbb{P}(W))$ of probability measures on $\mathbb{P}(W)$ such that, for all g in G, for ν-almost all x in X, we have $\rho_{gx} = \sigma(g, x)_*\rho_x$. According to [35], the quotient space $L\backslash\mathcal{P}(\mathbb{P}(W))$ has a countable basis. Consequently, by ergodicity, the map $x \mapsto \rho_x$ essentially takes its values in a single L-orbit of the form L/L_W where L_W is the stabiliser of a probability measure of $\mathbb{P}(W)$ in L. In other words, the cocycle σ is cohomologous to a cocycle σ_W with coefficients in L_W and in particular, by definition of L, L_W is Zariski dense in L (in fact, we can even show that $L_W = L$ but we don't need this). In particular, L_W acts irreducibly on W.

According to Lemma 4.5, there exists a decomposition $W = W_1 \oplus \cdots \oplus W_k$ of W as direct product and Euclidean norms $\|\cdot\|_1, \dots \|\cdot\|_k$ on $W_1, \dots, W_k$ such that, if M_W is the subgroup of L consisting of elements in L, which for $1 \le i \le k$ preserve W_i by acting in a conformal manner, then $L_W \cap M_W$ is finite index in L_W. If $M_W = L_W$, we have finished the analysis of the subquotient of W. In the general case, it is at this place that we need to introduce a finite extension of X. We put $\tilde{X}_W = X \times L_W/M_W$ and we provide it with a skew action associated to the cocycle σ_W. Then, by construction, the cocycle σ_W, and thus the cocycle σ, seen as cocycle over $\tilde{X}_W$, is cohomologous to a cocycle with coefficients in M_W.

The result now follows by defining $\tilde{X}$ to be the product fibre over X of all the systems $\tilde{X}_W$, where W varies among the set of irreducible subquotients of L in $\mathbb{R}^d$ □

4.4. **Osseledets Theorem and Lyapunov exponents.** In the previous paragraph, we developed a structure theorem for the cocycles above an action of an amenable group G. We are going to study now the phenomena that appear when G is $\mathbb{Z}$ or $\mathbb{R}$, because of the existence of order relations in these groups. To simplify the presentation and avoid difficult techniques, we will assume that $G = \mathbb{Z}$, the real case being analogous.

We thus give ourselves an ergodic automorphism $T : X \mapsto X$ of a Lebesgue space (X, ν). A cocycle $\sigma : \mathbb{Z} \times X \to H$ is then completely determined by the map $\sigma(1, \cdot)$.

In this paragraph, we will therefore talk about cocycles as maps $\sigma : X \to H$ when we use these maps to build automorphisms that are skew products $(x, y) \mapsto (Tx, \sigma(x)y)$. For such

[16]typo

a map, we write for x in X and n in $\mathbb{N}$.

$$\sigma_n(x) = \sigma(T^{n-1}x)\dots\sigma(x)$$

and, for integer $n \le -1$,

$$\sigma_n(x) = \sigma(T^{-n}x)^{-1}\dots\sigma(T^{-1}x)^{-1}.$$

We will systematically provide $\mathbb{R}^d$ with the usual Euclidean norm (the actual construction does not depend on this choice). We have immediately the

Lemma 4.7. *Let $\sigma : X \to \mathrm{GL}_d(\mathbb{R})$ be a cocycle. For all λ in $\mathbb{R}$, for all $x \in X$, the set*

$$V_\lambda(x) = \left\{ v \in \mathbb{R}^d | \limsup_{n\to\infty} \frac{1}{n} \log \|\sigma_n(x)v\| \le \lambda \right\}$$

is a vector subspace. We have $V_\lambda(Tx) = \sigma(x)V_\lambda(x)$.

In particular, the dimension of $V_\lambda(x)$ is an invariant function. By ergodicity, it is constant. We let d_λ be its value. The function $\lambda \mapsto d_\lambda$ is increasing.

Suppose henceforth that $\int_X \log\|\sigma(x)\| d\nu(x) < \infty$, so that, according to Birkhoff's theorem, $d_\lambda = d$ for λ sufficiently large, and also (but it would not be necessary) that $\int_X \log\|\sigma(x)^{-1}\| d\nu(x) < \infty$ such that, for the same reason, $d_\lambda = 0$ for sufficiently small λ. We call a cocycle $\sigma : X \to \mathrm{GL}_d(\mathbb{R})$ such that

$$\int_X \log \max(\|\sigma(x)\|, \|\sigma(x)^{-1}\|) d\nu(x) < \infty \tag{7}$$

to be integrable.

The increasing function with integer values $\lambda \mapsto d_\lambda$ has a finite number of discontinuities in the real numbers $\lambda_1 < \dots < \lambda_k$. We denote $0 < d_1 < \dots < d_k = d$ the values of the function at these discontinuities. The numbers $\lambda_1 < \dots < \lambda_k$ are called Lyapunov exponents of the cocycle σ. Write, for x in X and $1 \le i \le k$, $V_i(x) = V_{\lambda_i}(x)$. One says that the flag $V_1(x) \subsetneq \dots \subsetneq V_k(x)$ is the Lyapunov flag of the cocycle.

Let $H \subset \mathrm{GL}_d(\mathbb{R})$ be the stabiliser of a flag $\{0\} = W_1 \subsetneq \dots \subsetneq W_k = \mathbb{R}^d$ with $\dim W_i = d_i$, $1 \le i \le k$. Lemma 4.7 implies that σ is cohomologous to a cocycle with coefficients in H. We now apply Proposition 4.6 to the action of this cocycle on W_i/W_{i-1}.

Note that, in these reductions, the integrability property (7) still holds. Indeed, if Q is the stabiliser of a flag in $\mathrm{GL}_d(\mathbb{R})$, we have, by the Gram-Schmidt orthogonalisation procedure that $\mathrm{GL}_d(\mathbb{R}) = O(d)Q$, so that if σ is a cohomologous cocycle with coefficients in Q we can always write for x in X

$$\sigma(x) \in \varphi(Tx)Q\varphi(x)^{-1}$$

where φ takes values in $O(d)$ so that the integrability properties of σ are preserved.

Similarly, if now σ is cohomologous to a conformal cocycle, there exists a map $\varphi : X \to \mathrm{GL}_d(\mathbb{R})$ and a function $\gamma : X \to \mathbb{R}$ such that, for ν-almost all x in X,

$$\sigma(x) \in e^{\gamma(x)}\varphi(Tx)O(d)\varphi(x)^{-1}.$$

Write $\varphi(x) = e^{\theta(x)}\varphi_1(x)$ with $\det\varphi_1(x) = 1$ and we put $\gamma_1(x) = \gamma(x) + \theta(Tx) - \theta(x)$. We still have

$$\sigma(x) \in e^{\gamma_1(x)}\varphi_1(Tx)O(d)\varphi_1(x)^{-1}$$

and in particular

$$\gamma_1(x) = \frac{1}{d}\log|\det\sigma(x)|,$$

so that γ_1 is an integrable function.

It should be noted that this reduction does not disturb the study of the exponential behaviour of the vector norms in the linear skew action because of

Lemma 4.8. *Let $\sigma : X \to \mathrm{GL}_d(\mathbb{R})$ be an integrable cocycle. Suppose that σ is cohomologous to a cocycle with coefficients in $O(d)$. Then for ν-almost all x in X, for all v in $\mathbb{R}^d$, one has*

$$\frac{1}{n}\log\|\sigma_n(x)v\| \to_{n\to\infty} 0.$$

Proof. We put for n in $\mathbb{N}$, $f_n = \log\|\sigma_n\|$. Then, one has $f_{m+n} \le f_n \circ T^m + f_m$. According to Kingman's subadditive ergodic theorem [32], for ν-almost all x in X, the sequence $\frac{1}{n}f_n(x)$ converges to a limit ℓ independent of x. Let $\varphi : X \to \mathrm{GL}_d(\mathbb{R})$ and $\theta : X \to \mathrm{GL}_d(\mathbb{R})$ such that, for ν-almost all x in X one has $\sigma(x) = \varphi(Tx)\theta(x)\varphi(x)^{-1}$. It follows $\sigma_n(x) = \varphi(T^n x)\theta_n(x)\varphi(x)^{-1}$ and thus there exists an infinite sequence of integers n for which $f_n(x)$ remains bounded. It follows $\ell = 0$.

By reasoning the same way, we show that for ν-almost all x, $\frac{1}{n}\log\|\sigma_n(x)^{-1}\|$ tends to 0 as n goes to infinity. The result follows. □

We then have the

Theorem 4.9 (Osseledets)**.** *Let $\sigma : X \to \mathrm{GL}_d(\mathbb{R})$ an integrable cocycle and let us keep the notations introduced above. There are measurable families $x \mapsto W_i(x) \subset V_i(x)$ of vector subspaces of dimension $d_i - d_{i-1}$, $1 \le i \le k$ defined on X such that, for ν-almost all x, $W_i(Tx) = \sigma(x)W_i(x)$ and $V_i(x) = W_i(x) \oplus V_{i-1}(x)$. For all v in $W_i(x)$, one has*

$$\frac{1}{n}\log\|\sigma_n(x)v\| \to_{n\to\infty} \lambda_i \text{ and } \frac{1}{n}\log\|\sigma_n(x)v\| \to_{n\to-\infty} -\lambda_i.$$

Remark 4.10. In particular, for all $v \in V_i(x) \setminus V_{i-1}(x)$, one has

$$\frac{1}{n}\log\|\sigma_n(x)v\| \to_{n\to\infty} \lambda_i$$

Remark 4.11. The Lyapunov exponents of the cocycle $(\sigma\circ T^{-1})^{-1}$ over the automorphism T^{-1} are $-\lambda_k < -\lambda_{k-1} < \cdots < -\lambda_1$ and its Lyapunov flag at $x \in X$ is

$$W_k(x) \subsetneq W_{k-1}(x) \oplus W_k(x) \subsetneq \cdots \subsetneq W_1(x) \oplus \cdots \oplus W_k(x) = \mathbb{R}^d.$$

In particular, these families W_i are unique.

Remark 4.12. The Lyapunov flag depends only on the future of the trajectory. More precisely, let $\pi : (X, \nu, T) \to (Y, \xi, S)$ be a factor of (X, ν, T), that is to say that (Y, ξ) is a Lebesgue space, π is a measurable map that sends ν to ξ, $S : Y \to Y$ is measurable that preserves ξ and that $\pi T = S\pi$. We do not necessarily assume that S is invertible. Then, if σ is defined on Y, that is, if σ is of the form $\tau \circ \pi$ with $\pi : Y \to \mathrm{GL}_d(\mathbb{R})$, the spaces V_i, $1 \le i \le k$ are also defined over Y. On the other hand, the construction of the additional W_i uses the past of the dynamics, and these are not generally defined on Y. This will pose often difficulties in the practical applications of this theorem, as we will see later.

The proof of Theorem 4.9 is essentially based on Proposition 4.6 and on the

Lemma 4.13. *Let $\sigma : X \to \mathrm{GL}_d(\mathbb{R})$ be an integrable cocycle. We assume that σ is of the form*

$$\sigma = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A is of size (r,r), B is of size $(r,d-r)$ and C is of size $(d-r,d-r)$, for an integer $1 \leq r \leq d-1$. We suppose further that there are reals λ,μ with $\lambda+\mu<0$, and, for ν-almost all x in X,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A(x)^{-1} \dots A(T^{n-1}x)^{-1}\| \leq \lambda$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \|C(T^{n-1}x) \dots C(x)\| \leq \mu.$$

Let $U = \mathbb{R}^r \times \{0\}$. Then, there exists a measurable family of vector subspaces $x \mapsto V(x)$ of dimension $d-r$ of $\mathbb{R}^d$ such that, for ν-almost all x, one has $\mathbb{R}^d = U \oplus V(x)$ and $\sigma(x)V(x) \subset V(Tx)$.

Proof. For ν-almost all x, we let $S(x)$ the matrix defined by

$$S(x) = -\sum_{n=0}^{\infty} A(x)^{-1} \dots A(T^n x)^{-1} B(T^n x) C(T^{n-1}x) \dots C(x)$$

(which converges by assumption), so that

$$A(x)S(x) + B(x) = S(Tx)C(x)$$

and so

$$\begin{pmatrix} A(x) & B(x) \\ 0 & C(x) \end{pmatrix} \begin{pmatrix} S(x) \\ 1 \end{pmatrix} = \begin{pmatrix} S(Tx) \\ 1 \end{pmatrix} C(x).$$

In particular, if $p(x)$ denotes the matrix operator

$$\begin{pmatrix} 0 & S(x) \\ 0 & 1 \end{pmatrix},$$

$\sigma(x)$ sends the image of $p(x)$ to the image of $p(Tx)$, which finishes the proof. □

Proof. By replacing σ by a cohomologous cocycle, we can assume that the flag $V_1 \subsetneq \dots \subsetneq V_k = \mathbb{R}^d$ is constant. As explained above, the integrability property (7) is preserved under this transformation.

For $1 \leq i \leq k$, let us now apply Proposition 4.6 to the cocycle induced by σ in V_i/V_{i-1}. After replacing (X,ν,T) by a finite extension $(\tilde{X},\tilde{\nu},\tilde{T})$ and σ by a cohomologous cocycle, one can assume that, for a certain basis of V_i/V_{i-1}, adapted to the flag

$$V_{i-1} = V_{i0} \subsetneq \dots \subsetneq V_{i\ell_i} = V_i$$

for ν-almost all x, the matrix $\sigma(x)$ is of the form,

$$A_i(x) = \begin{pmatrix} A_{i1}(x) & * & \dots \\ 0 & \dots & \dots \\ 0 & \dots & A_{i\ell_i}(x) \end{pmatrix},$$

with $A_{ij}(x) = e^{\lambda_{ij}(x)} U_{ij}(x)$ where $\lambda_{ij}(x) \in \mathbb{R}$, $U_{ij}(x) \in O(r_{ij})$, $1 \leq j \leq \ell_i$. One can further assume that the functions λ_{ij} are integrable. Then, according to Birkhoff's theorem, one has

$$\frac{1}{n} \log \|A_{ij}(x)^{-1} \dots A_{ij}(T^{n-1}x)^{-1}\| \to -\int_{\tilde{X}} \lambda_{ij} d\tilde{\nu}$$

and

$$\frac{1}{n} \log \|A_{ij}(T^{n-1}x) \dots A_{ij}(x)\| \to \int_{\tilde{X}} \lambda_{ij} d\tilde{\nu}$$

so that, even if we apply Lemma 4.13 several times, we can assume that we have

$$\int_{\tilde{X}} \lambda_{i1} d\tilde{\nu} \leq \dots \leq \int_{\tilde{X}} \lambda_{i\ell_i} d\tilde{\nu}.$$

Then, for $1 \leq j \leq \ell_i$, for all vectors $v \in V_{ij} \setminus V_{i(j-1)}$ one has

$$\frac{1}{n} \log \|\sigma_n(x)v\| \to_{n\to\infty} \int_{\tilde{X}} \lambda_{ij} d\tilde{\nu}$$

and thus, by definition on the flag (V_i),

$$\int_{\tilde{X}} \lambda_{i1} d\tilde{\nu} = \dots = \int_{\tilde{X}} \lambda_{i\ell_i}{}^{17} d\tilde{\nu} = \lambda_i.$$

It follows that for all v in $V_i \backslash V_{i-1}$,

$$\frac{1}{n} \log \|\sigma_n(x)v\| \to_{n\to\infty} \lambda_i.$$

Finally, we have now for ν-almost every x in X

$$\frac{1}{n} \log \|A_i(x)^{-1} \dots A_i(T^{n-1}x)^{-1}\| \to \lambda_i$$

and

$$\frac{1}{n} \log \|A_i(T^{n-1}x) \dots A_i(x)\| \to -\lambda_i{}^{18}.$$

Lemma 4.13 can be applied multiple times to the transformation T^{-1} and to the cocycle $(\sigma \circ T^{-1})^{-1}$ to conclude the existence of the complements W_i.

□

5. Action of the triangular group

Let's denote from now on

$$A = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \middle| t \in \mathbb{R} \right\}$$

and

$$N = \left\{ n_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \middle| s \in \mathbb{R} \right\}.$$

We return to the proof of Theorem 3.4. We have an action of the group $P = AN$ on a probability space (X, ν), namely the stratum $\mathcal{H}(\alpha)$, equipped with an invariant ergodic probability measure. We are trying to show that this measure is actually $SL_2(\mathbb{R})$-invariant and affine. For this, like Ratner's theorem [29], the strategy will entail to use the geometry of the space (essentially, the flat structure constructed in Section 2), to show that the measure ν has additional invariance properties. Specifically, we will show that the conditional measures of ν along the foliation of X by affine subvarieties are invariant under unipotent affine transformations of bigger and bigger leaves. These leaves are invariant under the N-action which act by affine unipotent transformations and the first of these unipotent groups that appears is N itself. By a drifting process, the size of the unipotent group that preserves the conditional measures, until it has a property of maximality. Then, we exploit this property of maximality to refine the information of the conditional measures of ν (as in [[23], Sect 10]). These information will notably imply that the measure ν is $SL_2(\mathbb{R})$-invariant.

[17] typo
[18] check signs

In the second step, we use the fact that ν is $\mathrm{SL}_2(\mathbb{R})$-invariant and the properties of the conditional measures of ν referred to above will show that ν is affine.

Let's go back to the first step for a moment. We have an action of $P = AN$ on X which preserves a measure ν. As we have already mentioned, we will try to follow the orbits of points x and y that are very close in X until we obtain points x' and y' which are at macroscopic distance from each other. We will then need to understand the vector which in an affine chart allows us to go from x' to y'. For this we need to have a good knowledge of the evolution of some linear cocycles above the action of P on X.

For this purpose, Eskin and Mirzakhani develop in [[11], Sect 8-10] a number of new concepts for the cocycles over an action of P. We will try to give a quick presentation.

5.1. **Inert subspaces of the Lyapunov decomposition.** Assume we have a probability Lebesgue space (X, ν), equipped with an action of the group $P = AN$ (which is solvable, therefore amenable) with discrete stabilisers.

Let's start by constructing partitions adapted to the study of the action. Recall that a partition ξ of a Lebesgue space X in measurable sets is said to be a measurable partition if the quotient space $\xi\backslash X$, equipped with the image measure ν is still a Lebesgue space. We identify partitions that differ over a set of zero measure.

Given two (classes of) measurable partitions ξ and η, we say that ξ is finer than η and write $\eta \prec \xi$ if the atoms of ξ are contained in those of η, that is, if $\xi(x) \subset \eta(x)$ for ν-almost every x in X.

If ξ is a measurable partition of X, for t in $\mathbb{R}$, we denote $a_t\xi$ the partition $x \mapsto a_t\xi(a_{-t}x)$. The partition ξ is said to be invariant along A if $\xi \prec a_t\xi$ for all $t \geq 0$. One says that ξ is subordinate to the action of N, if, for ν-almost every x in X, we have $\xi(x) \subset Nx$, and the inverse image of $\xi(x)$ in N under the orbit map is a bounded set that contains the base point in its interior.

Remark 5.1. It is usually said that a measurable partition ξ of a Lebesgue space (X, ν) is invariant by an automorphism T if, for ν-almost every x in X one has $\xi(x) \subset T^{-1}\xi(Tx)$. I adopt here the inverse convention to respect the notations of the authors.

Lemma 5.2. *There is a measurable A-invariant subordinate partition with respect to the action of N.*

Proof. This statement is an adaptation of the existence of a Markov partition for hyperbolic dynamics. In this form, it is a direct generalisation of [[23], Prop 9.2] or [[11], Lem B1]. ☐

We will provide the atoms of such a partition with a probability measure induced by the Haar measure of N which we denote by ν_x^ξ.

The theory recalled in section 4[19], and in particular paragraph 4.3 applies to the group action of P. Following [[11], Sect 8-10], we will apply this theory to the tangent cocycle that arises when acting on the strata of Abelian differentials.

Suppose now (X, ν) is equipped with an action of P and let $\sigma : P \times X \to \mathrm{GL}_d(\mathbb{R})$ be a cocycle. The restriction of σ to A defines a linear cocycle σ_A over a dynamical system (X, T, ν) to which we can apply Osseledets theory of paragraph 4.4. We will study how N acts on the decomposition of $\mathbb{R}^d$ associated with this cocycle in Theorem 4.9 (we will henceforth assume that the cocycle $(t, x) \mapsto \sigma(a_t, x)$ is integrable). For x in X and $y = px$ in Px, we will sometimes write $\sigma(x, y)$ for $\sigma(p, x)$.

[19]french

For controlling the action of N locally, we introduce the assumption (which will be verified for the systems we are interested in) that the cocycle σ does not vary over local stable leaves. In more precise terms, we choose a measurable partition ξ which is A-invariant and subordinate to N and we assume that for ν-almost every x in X, for ν_x^ξ-almost every y in ξ we have $\sigma(x, y) = e$.

Write then $\lambda_1 > \cdots > \lambda_k$ the Lyapunov exponents of σ_A and for x in X, $V_1(x) \subsetneq \cdots \subsetneq V_k(x)$ the Lyapunov flag of σ_A at x. Note that, as remarked in Section 4.4, since σ_a is ξ-measurable and ξ is A-invariant, the families $x \mapsto V_i(x)$ are ξ-measurable.

Remark 5.3. To always respect the convention of the authors, we call the the Lyapunov flag the flag that in paragraph 4.4 has been designated as the Lyapunov flag of the cocycle $x \mapsto \sigma(a_{-1}, x)$ above the dynamics of $T = a_{-1}$.

On the other hand, let $x \mapsto W_i(x)$, $1 \le i \le k$ the families of vector subspaces defined by Theorem 4.9 applied to σ_A, so that W_i is the complement of V_{i-1} in V_i. A priori, there is no reason for the maps $x \mapsto W_i(x)$ to be ξ-measurable. For x in X, we define $E_i(x)$ the largest subspace of $\mathbb{R}^d$ such that we have $\sigma(x, y)E_i(x) \subset W_i(y)$ for almost all y in $\xi(x)$ (this space may be very well reduce to $\{0\}$ except for E_1, which is equal to V_1). We say, like Eskin and Mirzakhani, that $E_1(x), \ldots, E_k(x)$ are the inert Lyapunov spaces of the cocycle σ at x and $E(x) = E_1(x) \oplus \cdots \oplus E_k(x)$ is the inert subspace of the cocycle.

We check the

Lemma 5.4 ([11], Lem 8.3)**.** *The sequence of inert subspaces $E_1, \ldots, E_k$ do not depend on the choice of partition ξ.*

Let's now formulate intermediate result of [11] which plays a role in the inert space, as part of the first stage of the proof of the Theorem 3.4. We have an action of P on (X, ν) and we try to show that the conditional measures of ν along some foliations are invariant. To do this, as in the proof of Ratner's theorem [29], we will need to be able to construct from two points x close to y, the points x' and y' in their orbit under P, which are macroscopic distance from each other. We will also need good understanding of the vector passing from x' to y' in an affine chart. The following proposition will show that this vector tends to be close to an inert space.

Proposition 5.5 ([11], Prop 8.5)**.** *Suppose all the exponents of the Lyapunov cocycle σ_A are > 0. Then there exists $\alpha > 0$ such that, for all $\delta > 0$, there exists a measurable set $K \subset X$ of measure $\ge 1 - \varepsilon$ and $L_0 > 0$ such that for all x in K, v in $\mathbb{R}^d$ and $L > L_0$ there exists $L < t < 2L$ such that for all y in a subset of $\nu_{a_{-t}x}^\xi$-measure $\ge 1 - \delta$ of $\xi(a_{-t}x)$ for which we have*

$$d\left(\frac{\sigma(a_{s(t,y)}, y)v}{\|\sigma(a_{s(t,y)}, y)v\|}, E(a_{s(t,y)}y)\right) \le e^{-\alpha t},$$

where $s(t, y)$ is the largest real s such that $\|\sigma(a_s, y)v\| \le \|v\|$.

In other words, we start the dynamics of A at time $-t$ from x; the vector v moves along $\sigma(a_{-t}, x)v$ which tends to be small since the Lyapunov exponents of σ_A are > 0. Then we disturb $a_{-t}x$ a little, moving it by an element of n in N such that $y = nx$ belongs to $\xi(a_{-t}x)$ and then, finally we restart the dynamics in the other direction for time s until the vector $\sigma(a_s n a_{-t}, x)v$ is of macroscopic size. Thus, for most n, this macroscopic vector is very close to $E(x)$.

5.2. **Synchronised exponents.** According to Proposition 5.5, during the operations that we will carry out inside the orbits, the vectors we are following tend to go down in the inert direction, that is, we control their position in relation to the Osseledets decomposition. We now wish to clarify this information by studying the position of these vectors in relation to the decomposition resulting from the Proposition 4.6, applied to the spaces $E_i(x)$.

After replacing the system (X,ν) by a finite extension, this proposition provides for $1 \leq i \leq k$, an equivariant flag

$$\{0\} = E_{i0}(x) \subsetneq \cdots \subsetneq E_{i\ell_i}(x) = E_i(x)$$

and for $1 \leq j \leq \ell_i$, a scalar cocycle $\lambda_{ij} : P \times X \to \mathbb{R}$ such that on $E_{ij}(x)/E_{i(j-1)}(x)$, the action of σ is according to the exponent $e^{\lambda_{ij}}(x)$[20].

Note that, since σ is ξ-measurable, we can suppose that it is the same for λ_{ij}, that is to say that for ν-almost every x in X, for ν_x^ξ-almost every y in $\xi(x)$, we have $\lambda_{ij}(x,y) = 0$.

We now assume, as in Proposition 5.5, that the Lyapunov exponent $\lambda_1 > \cdots > \lambda_k$ of the cocycle σ_A are > 0, so that, according to Theorem 4.9, for all i,j, we have

$$\int_X \lambda_{ij}(a_t,x)d\nu(x) = \lambda_i > 0 \text{ for all } t > 0.$$

We will describe in this section an equivalence relation called synchronisation on the ξ-measurable and integrable cocycles $\lambda : P \times X \to \mathbb{R}$ of integral > 0, that is, such that

$$\int_X \lambda_{ij}(a_t,x)d\nu(x) > 0 \text{ for all } t > 0.$$

These constructions are carried out in [[11], Sect 10]. The motivation for the definition of this equivalence relation can be illuminated by the following example.

Example 5.6. Given a probability measure μ on $\mathbb{R}$ with compact support and is symmetric (different from the Dirac mass at $\{0\}$). Let $(X_n)_{n\geq 1}$ a sequence of independent random variables and law ν. We put $S_n = X_1 + \cdots + X_n$ and write M_n for the sequence of random matrices

$$M_n = \begin{pmatrix} e^{1+S_n} & 0 \\ 0 & e^{1-S_n} \end{pmatrix}.$$

For $\varepsilon > 0$, we write $n(\varepsilon)$ for the random integer which is the largest integer n such that the vectors $v_n(\varepsilon) = M_n \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix}$ is of norm ≤ 1.

We can easily convince ourselves that the two components of $v_n(\varepsilon)$ do not become big at the same time, that is, with probability tending to 1 when $\varepsilon \to 0$, the vector $v_{n\varepsilon}(\varepsilon)$ is close to one of the coordinate axes.

Eskin and Mirzakhani establish an analogous result for sequences of the form $\sigma(a_n, ux)v$ where v is a small vector and u plays the role of randomness in the previous example. They show that for a certain amount of u, these vectors are getting closer towards the union of vector spaces of $E(a_n ux)$. These subspaces are defined by bundling the subspaces of $E_{ij}(x)$ precisely when the cocycles λ_{ij} are synchronised.

Now define the following relationship. Let $\lambda : P \times X \to \mathbb{R}$ be a ξ-measurable cocycle such that λ_A is integrable and of integral > 0. For x in X we put

$$\mathcal{F}_\lambda[x] = \{y \in Px | \lambda(x,y) = 0\}.$$

[20]French. Orginal: l'action de σ soit conforme d'exposant $e_{ij}^{\lambda}(x)$

We will stay that two cocycle λ and μ such that λ_A and μ_A are integrable and of integral 0 are synchronised if there exist cocycles λ' and μ' cohomologous to λ and μ such that for ν-almost every x in X, we have
$$\mathcal{F}_{\lambda'}[x] = \mathcal{F}_{\mu'}[x].$$
This does not agree with the original definition of [[11], Sect 10], Let's briefly recall that these are equivalent.

Let λ like above. We may replace λ with a cohomologous cocycle, and assume that there exists $\epsilon > 0$ such that for all $t > 0$, for ν-almost every x in X, we have $\lambda(a_t, x) \geq \epsilon t$. Such a cocycle defines a time change for the flow A. More precisely, for u in $\mathbb{R}$ and for x in X, let $\tau_\lambda(u, x)$ be the unique real number such that
$$\lambda(a_{\tau_\lambda(u,x)}, x) = u$$
and $a_u^\lambda x = a_{\tau_\lambda(u,x)} x$. Then the cocycle relation implies that the family (a_u^λ) is a flow on X. Then note that it does not necessarily preserve ν but it acts absolutely continuously. This flow is the reparametrisation of the flow (a_t) along which the cocycle λ simply becomes a coordinate of time.

Since λ is ξ-measurable, for all $u \leq 0$, $\tau_\lambda(u, \cdot)$ is constant on $\xi(x)$ and thus the partition ξ is invariant by the flow (a_t^λ). For $u \geq 0$ and for x in X, we define $\mathcal{F}_\lambda[x, u]$ to be the set of $y's$ in Px for which $\lambda(x, y) = 0$ and such that $\xi(a_{-t}x) = \xi(a_{-t'}y)$ for some t and t' such that $\lambda(a_{-t}, x) \geq -u$ and $\lambda(a_{-t'}, y) \geq -u$[21]. We put
$$\mathcal{F}_\lambda[x] = \bigcup_{u \geq 0} \mathcal{F}_\lambda[x, u] = \{y \in Px : \lambda(x, y) = 0\},$$
which we'll see as the piece of the strong unstable leave of x for the flow (a_u^λ) and the orbit Px of x. We equip $\mathcal{F}_\lambda[x, u]$ with the measure $\nu_{x,u}^\xi$ which is the image under a_u^λ of the measure $\nu_{a_{-u}^\lambda x}^\xi$.

As always, let $\mu : P \times X \to \mathbb{R}$ another ξ-measurable cocycle such that μ_A is integrable and of integral > 0. Then λ and μ are synchronised if, for ν-almost every x in X, there exists $C > 0$ and $\theta > 0$ such that, for u sufficiently big, one has $|\mu(x, y)| \leq C$ for y in a set of $\nu_{x,u}^\xi$-measure $\geq \theta$ in $\mathcal{F}_\lambda[x, u]$. It is possible to show that one can choose θ arbitrarily close to 1.

Remark 5.7. The equivalence between these two definitions can be seen as an analogue of the following general fact: If (X, T, ν) is an ergodic dynamical system and if f is a real integrable function on X, then f is cohomologous to 0 (that is, f writes as $g \circ T - g$ for some function g) if and only if there exists $C > 0$ and $\theta > 0$ such that, for ν-almost every x in X, one has
$$|f(x) + \cdots + f(T^n x)| \leq C$$
for n in a set of integers with density $\geq \theta$.

5.3. **Bounded subspaces.** Let's continue to follow the constructions of [[11], Sect 10]. We introduce now the last abstract notion that will be useful in the study of linear cocycle over an ergodic action of P, and more precisely for the study of these cocycle in the inert components E_i, in connection with Zimmer's decomposition.

In fact, this decomposition produces a triangular cocycle with corresponding blocks of the form (6).

[21] Typo. Original: $\lambda(a_{-t}, x) \geq -u$ and $\lambda(a_{-t}, x) \geq -u$

Suppose for simplicity that we have a block of size $(p+q)\times(p+q)$

$$\begin{pmatrix} e^{\lambda}U & B \\ 0 & e^{\mu}V \end{pmatrix}$$

where U and V are cocycle with values in $O(p)$ and $O(q)$, λ and μ are scalar cocycles, integrable and of integral >0 and B is a matrix-valued function on $P\times X$.

Then the behaviour of the vector norm $\sigma(g,x)v$, where g is a large element of P and v is a vector in $\{0\}\times\mathbb{R}^q$, is not necessarily controlled by the cocycle $\mu(g,x)$. We will introduce a subspace of $\{0\}\times\mathbb{R}^q$ where this is the case.

Let us take up again the notations of the preceding paragraph for ξ,σ,λ_{ij}, etc.

Let $1\le i\le k$ and $1\le j\le \ell_i$. For x in X, we write $\mathcal{F}_{ij}[x]$ for $\mathcal{F}_{\lambda_{ij}}[x]$. We define the bounded subbundles $E_{ij,\mathrm{bdd}}\subset E_{ij}$ as the largest measurable family $x\mapsto E_{ij,\mathrm{bdd}}(x)$ of subspaces of $E_{ij}(x)$ which are σ-equivariant (that is, such that $E_{ij,\mathrm{bdd}}(gx)=\sigma(g,x)E_{ij,\mathrm{bdd}}(x)$ for all g in P and for ν-almost every x) and in which σ is cohomologous with a cocycle whose restriction to $\mathcal{F}_{ij}$ has values in a compact group. In other words, there is a measurable family $x\mapsto\varphi_{x,ij}$ of scalar products on $E_{ij,\mathrm{bdd}}(x)$ such that, for ν-almost every x in X, for all y in $\mathcal{F}_{ij}[x]$[22] for all v,w in $E_{ij,\mathrm{bdd}}(x)$ one has

$$\varphi_{y,ij}(\sigma(x,y)v,\sigma(x,y)w)=\varphi_{x,ij}(v,w)$$

and $E_{ij,\mathrm{bdd}}$ is the largest bundle with this property.

Remark 5.8. The existence of $E_{ij,\mathrm{bdd}}$ can be seen as a generalisation of the following fact: if (X,T,ν) is an ergodic dynamical system and $\sigma: X\to\mathrm{GL}_d(\mathbb{R})$ a cocycle, there is a largest σ-equivalent subbundle $x\mapsto V(x)$ of vector subspaces in $\mathbb{R}^d$ in which σ preserves a scalar product.

Remark 5.9. Again, this is not exactly the definition of $E_{ij,\mathrm{bdd}}$ given in [[11], Sect 10]. The equivalence between the definition still results from a similar reasoning to that described in Remark 5.7.

Finally, in the bundle E, we can gather the vectors for which the behaviour in asymptotics under the action of the cocycle is essentially of the same type: For $1\le i\le k$ and $1\le j\le\ell_i$, we define $E_{ij,\mathrm{bdd}}$ as the direct sum

$$\bigoplus_{pq\sim ij} E_{pq,\mathrm{bdd}}$$

where pq ranges among the set of pairs such that λ_{pq} and λ_{ij} are synchronised. We can then state the result of [11] which characterises in our situation the analogue of this phenomenon described in Example 5.6.

Proposition 5.10 ([11], Prop 10.1)**.** *Suppose all Lyapunov exponents of the cocycle σ_A are >0. Then, there exists $\theta>0$ and, for all $\delta>0$, there exists a measurable set $K\subset X$ of measure $\ge 1-\varepsilon$ and $L_0>0$ such that, for all x in K, v in $E(x)$ and $t>L_0$, for all y in a subset of $\xi(a_{-t}x)$ of $\nu^{\xi}_{a_{-t}x}$-measure $\ge\theta$, we have*

$$d\left(\frac{\sigma(a_{s(t,y)},y)v}{\|\sigma(a_{s(t,y)},y)v\|},\bigcup_{ij}E_{ij,\mathrm{bdd}}(a_{s(t,y)}y)\right)\le\delta,$$

where $s(t,y)$ is the largest real number s such that $\|\sigma(a_s,y)v\|\le\|v\|$.

[22]Typo. Original: y in $\mathcal{F}_{ij,bdd}[x]$

Thus Proposition 5.5 shows that the vectors that appear in the constructions belong to the inert subspace E. Proposition 5.10 allows to show that part of them belong to $E_{ij,\mathrm{bdd}}$.

5.4. **Triangular Action.** The objects we have just presented are those that appear in the beginning of the inductive process, which establishes the first stage of the proof of Theorem 3.4. In the induction, as we have already mentioned, we improve the invariance properties of the measure ν along some foliations. During the recurrence, it is then necessary to replace the orbits of the group N by higher-dimensional subvarieties of the stratum $\mathcal{H}(\alpha)$. These submanifolds are naturally parametrised by homogeneous spaces of simply connected nilpotent Lie groups. The fact that the measure ν was N-invariant is then replaced by the fact that the conditional measure of ν along one of these submanifolds is a Haar measure of a homogeneous space that parametrises them.

Let's describe this structure more precisely. Given a Lebesgue space (X,ν), we will call the triangular action on X (I invent this terminology) to be the data

(i) a measurable action preserving ν of the group $A = \{a_t | t \in \mathbb{R}\}$ on X,
(ii) a simply connected nilpotent group U and Lie algebra $\mathfrak{u}$
(iii) a cocycle $\theta : A \times X \to \mathrm{Aut}(U)$,
(iv) a measurable θ-equivariant family $x \mapsto U_x$ of closed connected subgroups of U
(v) a measurable map $\pi : U \times X \to X$ (which is not necessarily an action of U on X)

such that for ν-almost every x in X, for all t in $\mathbb{R}$, u in U and v in U_x,

$$a_t\pi(u,x) = \pi(\theta_{t,x}(u), x)$$
$$\pi(e,x) = x$$
$$\pi(uv,x) = \pi(u,x)$$
$$\pi(U,\pi(u,x)) = \pi(U,x)$$

and that

(i) for ν-almost every x in X, π induces an injection $U/U_x \to X$ and the conditional measure of ν along $\pi(U,x) \simeq U/U_x$ identifies with the U-invariant measure on U/U_x;
(ii) the Lyapunov exponents of the cocycle θ, seen as cocycle with values in the group $\mathrm{Aut}(\mathfrak{u}) \subset \mathrm{GL}(\mathfrak{u})$ are > 0.

The property regarding the conditional measures make sense because the axioms imply that the sets $\pi(U,x)$ are classes of a measurable equivalence relation on X.

This structure can be understood as follows: we have a partition of X in sets that are parametrised by homogeneous spaces of the nilpotent group U. These sets are a priori not the orbits of an action of U, but they are dilated by the action of (a_t). The conditional measures of ν of this partition a read like Haar measures. In particular, the larger group U, the larger measure ν have now invariance (in some foliations).

An action of the group P naturally defines such a structure

Remark 5.11. The sets that I have denoted $\pi(U,x)$ are denoted $U^+[x]$ in [[11],Sect 3-12] and are called generalised affine subspaces.

Example 5.12. Let's give a classic example in which such a structure appears naturally. Let $G = \mathrm{SL}_2(\mathbb{C})$, M the group of diagonal matrices with coefficients of modulus 1 in G, U the upper triangular unipotent group in G and Γ be a lattice in G. We are considering A as subgroup of G. Let $X = M\backslash G/\Gamma$ and let us be given the probability measure ν which is the image measure of a Γ-invariance probability measure on $Y = G/\Gamma$. The space Y is equipped with an action of AU in which A dilates the orbits of U. As M normalises U, the partition in U-orbits of Y induces a partition of X. The choice of a measurable section of the natural projection $Y \to X$ gives a parametrisation by U of atoms of this partition. On the other hand, these cannot be written as orbits of an action U on X.

The set of concepts introduced in this part for the actions of $P = AN$ (invariant partitions, cocycles, Osseledets decomposition, Zimmer's reduction, synchronised cocycles, bounded subspaces) can be developed within the framework of triangular actions: this is the object of [[11], Sect3-4, 8-10]. In a proof by induction, which makes it possible to prove [[11], Theorem 2.1], one replaces the group U by a supergroup. We will now try to give more information on this induction.

6. The proof of Eskin-Mirzakhani

We are leaving the abstract framework of section 5 and return to the study of the action of P and of $\mathrm{SL}_2(\mathbb{R})$ on the stratum $X = \mathcal{H}_1(\alpha)$, equipped with a flat structure modelled on $H^1(S, \Sigma, \mathbb{R}^2)$, introduced in section 2.

6.1. **A Theorem of Forni.** An essential point of the dynamical properties of these action lies in the possibility to define a strong unstable lamination for the action of A only in terms of this flat structure. This stems from a remarkable result of Forni that we are going to state.

Recall that the flat structure of $\mathcal{H}(\alpha)$ allows to build a number of $\mathrm{GL}_2(\mathbb{R})^+$-equivariant vector bundles on $\mathcal{H}_1(\alpha)$, as explained in section 2.4, and in particular the Hodge bundle $H^1(S, \mathbb{R})$ and its subbundle $p(H^1_\perp)$ of codimension 2.

Given a subgroup H of $\mathrm{SL}_2(\mathbb{R})$ and a H-invariant probability measure ν on $\mathcal{H}_1(\alpha)$, the choice of a measurable trivialisation of the Hodge bundle defines a cocycle $H \times X \to \mathrm{GL}(H^1(S, \mathbb{R}))$. By construction, this cocycle takes its values in the symplectic group of the form $\overline{\omega}$ induced by the cup product (see paragraph 2.2). The choice induced by another trivialisation produces a cohomologous cocycle. This cocycle (or rather its cohomologous class) is called the Kontsevich-Zorich cocycle in the literature and was introduced in [18].

The theorem of Forni describes the Lyapunov exponents:

Theorem 6.1. *(Forni, [13]) Let ν be a A-invariant ergodic probability Borel measure on $\mathcal{H}_1(\alpha)$. The Lyapunov exponents of the Kontsevich-Zorich cocycle with respect to ν are of the form*

$$\lambda_1 = 1 > \lambda_2 \geq \cdots \geq \lambda_g \geq 0 \geq -\lambda_g \geq \cdots \geq -\lambda_2 > -\lambda_1 = -1$$

Remark 6.2. In this formula, we count the exponents with multiplicity.

The symmetry properties of the sequence of Lyapunov exponents are immediate because the cocycle preserves a symplectic form. The important information given by the theorem is that the first exponent is simple, that is, $\lambda_2 < 1$. It can be reformulated as follows: all the Lyapunov exponents of the A-action in the bundle $p(H^1_\perp)$ are < 1.

6.2. **Geometry of the unstable foliation.** We will describe here the lamination of $\mathcal{H}_1(\alpha)$ into affine subvarieties of $\mathcal{H}(\alpha)$. This lamination is A-equivariant and Forni's theorem allows to see it as the strong unstable foliation for the action of A.

Recall from section 2.4 in which we have decomposed the vector bundle $\mathcal{H}(\alpha) \times_\Gamma H^1(S, \Sigma, \mathbb{R})$ as sum of a bundle which is identified in a $\mathrm{GL}_2(\mathbb{R})^+$-equivariant way to $\mathcal{H}(\alpha) \times \mathbb{R}^2$ and of the bundle $H^1_\perp$. As the tangent bundle of $\mathcal{H}(\alpha)$ is the tensor product of $\mathbb{R}^2$ with $\mathcal{H}(\alpha) \times_\Gamma H^1(S, \Sigma, \mathbb{R})$, this gives a $\mathrm{GL}_2(\mathbb{R})^+$-equivariant decomposition

$$T\mathcal{H}(\alpha) \simeq (\mathcal{H}(\alpha) \times (\mathbb{R}^2 \otimes \mathbb{R}^2)) \oplus (\mathbb{R}^2 \otimes H^1_\perp). \tag{8}$$

Let $e_1 = (1,0)$ and $e_2 = (0,1)$ the canonical basis of $\mathbb{R}^2$. We check that in the decomposition (8), the tangent bundle of $\mathcal{H}_1(\alpha)$ is

$$T\mathcal{H}_1(\alpha) \simeq (\mathcal{H}_1(\alpha) \times (\mathbb{R}e_1 \otimes e_1) \oplus \mathbb{R}(e_1 \otimes e_2 + e_2 \otimes e_1) \oplus \mathbb{R}e_2 \otimes e_2) \oplus (\mathbb{R}^2 \otimes H^1_\perp). \tag{9}$$

In particular, let's introduce the subbundles

$$W^+ \simeq (\mathcal{H}_1(\alpha) \times (\mathbb{R}e_1 \otimes e_1) \oplus (\mathbb{R}e_1 \otimes H^1_\perp) \tag{10}$$

$$\text{and } W^- \simeq (\mathcal{H}_1(\alpha) \times (\mathbb{R}e_2 \otimes e_2) \oplus (\mathbb{R}e_2 \otimes H^1_\perp). \tag{11}$$

Then these subbundles are A-equivariant. The tangent of the action of N on the right is $\mathcal{H}_1(\alpha) \times (\mathbb{R}e_1 \otimes e_1) \subset W^+$. Finally, according to (9), we have the decomposition

$$T\mathcal{H}_1(\alpha) \simeq (\mathcal{H}_1(\alpha) \times \mathbb{R}(e_1 \otimes e_2 + e_2 \otimes e_1)) \oplus W^+ \oplus W^- \tag{12}$$

in which the 1-dimensional component is the tangent line of the A-action.

If the action of A in W^+ (resp. W^-) were uniformly expanding, (12) would imply that the flow A is Anosov (except that $\mathcal{H}_1(\alpha)$ is not compact). The theorem of Forni translates directly into a weak version of these expansion properties:

Corollary 6.3. Let ν be a Borel probability measure that is A-invariant and ergodic on $\mathcal{H}_1(\alpha)$. The Lyapunov exponents of A in W^+ (resp. W^-) with respect to ν are all > 0 (resp. < 0).

Proof. Recall that we have an exact sequence

$$0 \to H^0(\Sigma, \mathbb{R})/H^0(S, \mathbb{R})^{23} \to H^1(S, \Sigma, \mathbb{R}) \to_p H^1(S, \mathbb{R}) \to 0$$

thanks to which we can build an exact sequence of vector bundles over $\mathcal{H}(\alpha)$. The theorem of Forni implies that all the Lyapunov exponents of the bundle

$$p(H^1_\perp) \subset \mathcal{H}(\alpha) \times_\Gamma H^1(S, \mathbb{R})$$

are of modulus < 1. The same is true for the Lyapunov exponents of the vector bundle $\mathcal{H}(\alpha) \times_\Gamma H^0(\Sigma, \mathbb{R})$. Indeed, the space $H^0(\Sigma, \mathbb{R})$ is identified with the free vector space generated by Σ. In this space, the mapping class group $\Gamma_{\alpha,\Sigma}$ acts by permutation of elements of the base, and thus preserves a metric. By construction, the flat bundle $\mathcal{H}(\alpha) \times_{\Gamma_{\alpha,\Sigma}} H^0(\Sigma, \mathbb{R})$ has a $SL_2(\mathbb{R})$-invariant metric, and in particular, the Lyapunov exponents of A are all zero. By definition of W^+, the sequence of Lyapunov exponents of A, counted with multiplicity, is thus of the form

$$2 > 1 + \lambda_2 \geq \cdots \geq 1 + \lambda_g \geq 1 \geq \cdots \geq 1 - \lambda_g \geq \cdots \geq 1 - \lambda_2$$

(where 1 appears $|\Sigma| - 1 = n - 1$ times) and the numbers are all > 0. □

6.3. $SL_2(\mathbb{R})$**-invariance and conditional measures.** We will now state Theorem 2.1 of [11] that we have so far described as the first step in the proof of the main theorem 3.4.

For this purpose, let's proceed to some geometric constructions in terms of the vector bundle W^+ (resp. W^-). We check easily using the charts of the affine structure of $\mathcal{H}(\alpha)$ that the distribution W^+ is integrable. For x in $\mathcal{H}(\alpha)$ we write $W^+[x]$ for the associated sheet: It is an affine subvariety in direction $W^+(x)$ that contains the affine line Nx.

Given a probability measure ν on X, we can then build a measurable family $x \mapsto \nu_{W^+}(x)$ where, for ν-almost all x, $\nu_{W^+}(x)$ is a Radon measure on the sheet $W^+[x]$ (equipped with the topology of the sheet), such that for any measurable partition ξ of (X, ν) subordinate

[23] typo

to the foliation W^+, for almost all x in X, the conditional measure of ν on the atom $\xi(x)$ is proportional to the restriction of $\nu_{W^+}(x)$ to $\xi(x)$. This family is unique up to multiplication with a positive function. By abuse of language, we still call $\nu_{W^+}(x)$ the conditional measure of ν along $W^+[x]$.

Theorem 6.4 ([11],Thm 2.1)**.** *Let ν be a P-invariant and ergodic probability measure on $X = \mathcal{H}_1(\alpha)$. Then ν is $\mathrm{SL}_2(\mathbb{R})$-invariant and there is a measurable distribution defined ν-almost everywhere and $\mathrm{SL}_2(\mathbb{R})$-equivariant subspaces $x \mapsto \mathcal{L}(x)$ of the bundle $\mathcal{H}(\alpha) \times_{\Gamma_{\alpha,\Sigma}} H^1(S, \Sigma, \mathbb{R})$ such that, for ν-almost all x in X, the conditional measure $\nu_{W^+}(x)$ is a Lebesgue measure of the affine space*

$$x + (\mathbb{R}^2 \otimes \mathcal{L}(x) \cap W^+(x)) \subset W^+[x].$$

Note that, since the $\mathrm{SL}_2(\mathbb{R})$-measure and the distribution $\mathcal{L}$ are $\mathrm{SL}_2(\mathbb{R})$-equivariant, the theorem implies the symmetry property for ν-almost all $x \in X$, and the conditional measure $\nu_{W^-(x)}$ is the Lebesgue measure of the affine subspace

$$x + (\mathbb{R}^2 \otimes \mathcal{L}(x) \cap W^-(x)) \subset W^-[x].$$

In fact, during the proof, we start establishing this property of symmetry then we deduce the theorem.

More specifically, as we have already mentioned, the proof of Theorem 6.4 is based on a recurrence argument in which we improve little by little the invariance properties of ν along the W^+-lamination. This argument is contained in [[11], Prop 12.1]. Once these invariance properties are shown to be maximal in a certain sense, one uses this maximality property to also conclude that the conditional measures of ν along W^- have the same properties of invariance by an entropy argument inspired by [[23], Sect 10]

6.4. **Invariance under unipotent groups.** In this paragraph we shall state [[11], Prop 12.1].

Let's start by giving details on the symmetry properties that exists between the foliations W^+ and W^-. Fix an A-invariant and ergodic measure ν on X.

Let $\mathcal{L}^- \subset W^-$ be an A-equivariant distribution of vector subspaces, defined ν-almost everywhere. Then, as

$$W^- = (\mathcal{H}_1(\alpha) \times (\mathbb{R}e_2 \otimes e_2)) \oplus (\mathbb{R}e_2 \otimes H^1_\perp)$$

and that, according to Theorem 6.1, the Lyapunov exponents of A in $\mathbb{R}e_2 \otimes H^1_\perp$ are > -2, we have

$$\mathcal{L}^- \subset \mathbb{R}e_2 \otimes H^1_\perp \text{ or } \mathcal{L}^- = (\mathcal{H}_1(\alpha) \times \mathbb{R}e_2 \otimes e_2)) \oplus (\mathcal{L}^- \cap \mathbb{R}e_2 \otimes H^1_\perp).$$

In the first case we write $\mathcal{L}^- = \mathbb{R}e_2 \otimes \mathcal{L}_\perp$ and put $\mathcal{L}^+ = \mathbb{R}e_1 \otimes \mathcal{L}_\perp$. In the second case, we write

$$\mathcal{L}^- = (\mathcal{H}_1(\alpha) \times (\mathbb{R}e_2 \otimes e_2)) \oplus (\mathbb{R}e_2 \otimes \mathcal{L}_\perp)$$

and put

$$\mathcal{L}^+ = (\mathcal{H}_1(\alpha) \times (\mathbb{R}e_1 \otimes e_1)) \oplus (\mathbb{R}e_1 \otimes \mathcal{L}_\perp).$$

This correspondence establishes the bijection between the A-equivariant distributions of W^+ and those of W^-, which respect the symmetry appearing in (10).

Proposition 6.5 ([11], Prop 12.1)**.** *Let ν be a P-invariant ergodic measure on X. For ν-almost every x in X, let $\mathcal{L}^-(x)$ the direction of the affine subspace of $W^-[x]$ generated by the support of $\nu_{W^-}(x)$, that is to say that the smallest vector subspace of $W^-(x)$, such that*

ν_{W^-} is supported on $x + \mathcal{L}^-(x)$. Let $\mathcal{L}^+ \subset W^+$ as above. Then, ν-almost every x in X, the measure $\nu_{W^+}(x)$ is invariant under translations by elements in $\mathcal{L}^+(x)$.

This statement is not exactly that of [[11], Prop 12.1], but it is deduced from the beginning of [[11], Sect 13]. It is the analogue of a statement appearing in the proof of Ratner's theorem by Margulis and Tomanov [[23], Cor 8.3]. We deduce from [[11], Sect 13] that $\nu_{W^+}(x)$ and $\nu_{W^-}(x)$ are Lebesgue measures on the affine spaces $x + \mathcal{L}^+(x)$ and $x + \mathcal{L}^-(x)$ by entropy reasoning similar to that of [[23], Sect 10].

Remark 6.6. The fact that ν is N-invariant implies that the entropy of a_1 with respect to ν is > 0. We deduce that, necessarily, the conditionals ν_{W^-} are not Dirac masses, and that $\mathcal{L}^-$ is of dimension > 0. This entropy argument is a simplified version of that of [[11], Sect 13].

Let's now describe the structure of the proof of Proposition 6.5. It is a matter of showing that, if the support of $\nu_{W^-}(x)$ is too large, the measure $\nu_{W^+}(x)$ would possess invariance properties. The ideal would be to show that $\nu_{W^+}(x)$ is invariant under a group of translations of the affine space $W^+[x]$. Unfortunately, we do not arrive directly with this statement, but we start by showing that this measure has invariance properties under the action of unipotent groups of affine transformations of $W^+[x]$.

Let's explain exactly what affine transformations can emerge during this construction.

Let $2 = \lambda_1 > \cdots > \lambda_k > 0$ be the Lyapunov exponents of the action of A on W^+ with respect to ν (counted without multiplicity) and

$$\{0\} = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_k = W^+$$

the associated Lyapunov flag. Note that $V_1 = \mathbb{R}e_1 \otimes e_1$, and more generally, for $i \geq 1$, $W_i = \mathbb{R}e_1 \otimes e_1 \oplus W_{i,\perp}$ for some $W_{i,\perp} \subset H^1_\perp$. Like Eskin and Mirzakhani in [[11], Sect6], we note, for ν-almost every x in X, $Q_+(x)$ the group of elements g in $\mathrm{GL}(W^+(x))$ which preserve the Lyapunov flag at x and such that, $1 \leq i \leq k$, for all v in $W_i(x)$, $gv \in v + W_{i-1}(x)$ (in the language of algebraic groups, $Q_+(x)$ is the unipotent radical of the parabolic group that stabilises the Lyapunov flag). We denote by $\mathcal{G}_+(x)$ the group of affine automorphism of $W^+[x]$ whose linear part belongs to $Q_+(x)$. Note that since ν is N-invariant, we have $N \subset \mathcal{G}_+(x)$ for ν-almost every x in X (where we abusively identified N with the group of automorphisms it induces on $W^+[x]$). We will always consider $W^+(x) \subset \mathcal{G}_+(x)$ by identifying the vectors with the associated translations.

Before showing that $\nu_{W^+}(x)$ is a Lebesgue measure, we show that it possesses invariance properties by subgroups of $\mathcal{G}_+(x)$. More precisely, one constructs a measurable partition $\eta_{W^+[x]}$ of the sheet $W^+[x]$ and a connected closed subgroup $U^+(x)$ of $\mathcal{G}_+(x)$ (I do not know why the + goes from bottom to top, but I try to respect the conventions of the authors) such that, for ν-almost every x in X, the conditional measure $\nu_{W^+,\eta}(x)$ of $\nu_{W^+}(x)$ along $\eta_{W^+[x]}$[24] are $U^+(x)$. The family η of measurable partitions on the sheets $W^+[x]$ are equivariant under A (in the sense that $a_t\eta_{W^+[x]}(x) = \eta_{W^+[a_t x]}(a_t x)$ for t in $\mathbb{R}$ and for ν-almost every x in X), so we can assume that the family of groups $x \mapsto U^+(x)$ is A-equivariant. Proposition 6.5 is deduced from

Proposition 6.7 ([11], Prop 12.1)**.** *Let ν be a P-invariant and ergodic measure on X. Suppose we have a measurable A-equivariant family $\eta_{W^+[x]}$ of measurable partitions of the sheets $W^+[x]$ and an A-equivariant measurable family $x \mapsto U^+(x) \subset \mathcal{G}_+(x)$ for which, for*

[24] typo

ν-almost every x in X, we have $N \subset U^+(x)$ and that the conditional measure $\nu_{W^+,\eta}(x)$ of $\nu_{W^+}(x)$ along $\eta_{W^+[x]}(x)$ is $U^+(x)$-invariant.

Let $\mathcal{L}^-(x) \subset W^-(x)$ the vector space direction of the affine subspace of $W^-[x]$ generated by the support of $\nu_{W^-}(x)$ and $\mathcal{L}^+(x) \subset W^+(x)$ the vector subspace associated by the symmetry (10)*. Then, if $\mathcal{L}^+(x)$ is not almost surely contained in $U^+(x)$, there exists families η_{new} and $U^+_{\mathrm{new}}(x)$ having the same properties and such that for ν-almost every x in X, $U^+(x)$ is a proper subgroup of $U^+_{\mathrm{new}}(x)$.*

It is the presence of these nilpotent groups U^+ that made it necessary to introduce triangular actions discussed in paragraph 5.4.

Indeed, the equivariance of the family U^+ under the group A implies, by analogous arguments used to those in paragraph 4.2 (the quotient of an algebraic variety by the action of an algebraic group is second countable), that the groups $U^+(x)$ are always conjugated to the same group U. The property that the cocycle θ of paragraph 5.4 has all its Lyapunov exponents strictly positive implies that U^+ is a subdistribution of $\mathcal{G}_+(x)$ in which one knows by construction that all the Lyapunov exponents of A are strictly positive.

6.5. **The drift.** In this paragraph we will attempt to present the drift argument that allows to increase the size of the group U^+ in Proposition 6.7. It is based on a general principle, whose introduction seems to me due to Katok and Spatzier [17], which has been used in recent work on dynamical systems in homogeneous spaces [[7], [22], [8], [4]] and which we shall explain.

Suppose we are given an action of a group G on a Lebesgue probability space (X, ν). We take a partition $x \mapsto W[x]$ of X in measurable sets, where each of these sets are equipped with a geometric structure (in homogeneous spaces, these sets are the orbits of an action of a connected Lie group H; in the case of strata, they are affine spaces). We assume that for every g in G and for ν-almost every x in X, we have $W[gx] = gW[x]$, that is to say that G permutes the atoms of the partition, and we also assume that g preserves the geometric structure of these sets (in the case of homogeneous spaces, G normalises the group H; in the case of strata, $G = AN$ acts by a cocycle with coefficients in the affine group).

With this data, we can associate a family $x \mapsto \nu_W(x)$ of Radon measures over $W[x]$ (Radon measures in the sense of the intrinsic topology of $W[x]$, the underlying topology of the geometric structure). The $\nu_W(x)$ are defined by the property that, if ξ is a measurable partition subordinate to the partition W (that is, for ν-almost every x in X, $\xi(x)$ is relatively compact in $W^+[x]$ and contains x in its interior), for ν-almost every x in X, the conditional measure of ν on $\xi(x)$ is proportional to the restriction of $\nu_W(x)$. Note that the equivariance properties of the action of G imply that for g in G, for ν-almost every x in X, $\nu_W(gx)$ is proportional to $g_*\nu_W(x)$.

We want to show that ν_W has invariance properties related to the geometric structure of W (in case of homogeneous spaces, $\nu_W(x)$ is invariant under a subgroup H; in the space of strata, under a subgroup the affine group). For this, the approach is the following. We start by building, using a theorem of Lusin, a compact subset Y of X (if there is no topology, we add some that is compatible with the Borel structure), where the map $x \mapsto \nu_W(x)$ is continuous. We seek to construct sequences (x_n) and (y_n) in Y and (g_n) in G such that $d(x_n, y_n) \to 0$, $g_n x_n$ and $g_n y_n$ still belong to Y and tend to elements x_∞ and $y_\infty \in W[x_\infty]$ with $x_\infty \neq y_\infty$.

The the equivariance and continuity properties should allow to show that the conditional measure $\nu_W(x_\infty)$ posses invariance under a transformation of $W[x_\infty]$ which sends x_∞ to y_∞ (a group translation, an affine transformation).

To ensure that $g_n x_n$ and $g_n y_n$ belong to Y, we employ ergodic theorems of the G-action. The most difficult part is to ensure that both $g_n x$ and $g_n y_n$ belong to Y and the distance $d(g_n x_n, g_n y_n)$ remains bounded from above and below.

The method used by Eskin and Mirzakhani is inspired by that introduced in [4], in which they introduce new degrees of freedom which improves considerably flexibility. Let's outline their construction now. We return therefore to the case where X is the stratum $\mathcal{H}_1(\alpha)$, equipped with a P-invariant ergodicity probability measure ν. We assume that we have a measurable A-equivariant family $x \mapsto U^+(x)$ of affine transformations of $W^+[x]$, like in Proposition 6.7.

We then obtain points q and $q' \in W^-(q)$ with $q' \neq q$, which satisfy equidistribution properties, derived from Birkhoff's theorem for the flow A, which guarantee that the orbits $\{a_t q | t \geq 0\}$ and $\{a_t q' | t \geq 0\}$ spend a lot of time in compact sets of large measure where certain a priori measurable maps are continuous (the conditional measures, the Osseledets decomposition of the Kontsevich-Zorich cocycles, etc.). As noted in Remark 6.6, the fact that one can find two such points on the same sheet W^- stems from the fact that, as the measure is N-invariant, the entropy of A is > 0.

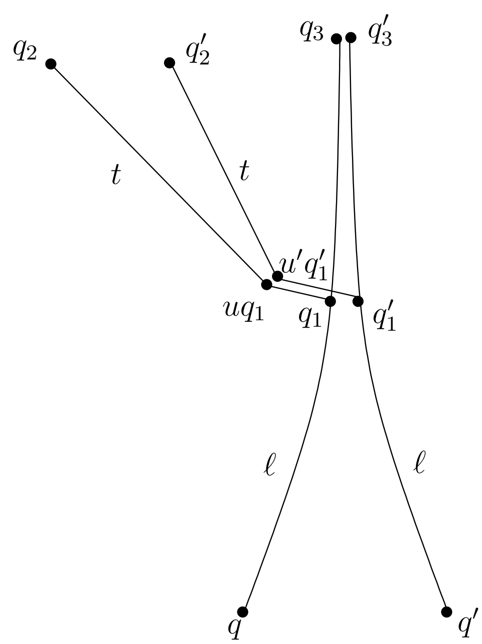

Figure 1. Construction of the drift

The flow is allowed to flow for some time ℓ (to be chosen later) and we obtain two points $q_1 = a_\ell q$ and $q_1' = a_\ell q'$. We then disturb q_1 (resp. q_1') by applying a small element u (resp. u') of $U^+(q_1)$ (resp. $U^+(q_1')$). We now have $u'q_1' \notin W^+(uq_1)$, so the flow tends to move these points away. We then launch the flow for some time τ until the points $q_2 = a_\tau u q_1$ and $q_2' = a_\tau u' q_1'$ [25] are at macroscopic distance from each other. Then arguments based on Proposition 5.5 and 5.10 allow to show that, for correctly chosen u and u', one can guarantee that in an affine chart containing q_2 and q_2', the vector $q_2' - q_2$ which allows to go from q_2

[25] typo

to q'_2 is in a direction very close to $E_{ij,\mathrm{bdd}}(q_2)$, for a certain (i,j) (we use the notation of section 5). Note that to apply this argument, it is necessary to be able to guarantee that q_1 and q'_1 belong to prescribed sets of measure close to 1: this is possible, provided that one can chose ℓ in a set of reals of density close to 1.

Similarly, as noted above in the general description of the method, we must be able to guarantee that the points q_2 and q'_2 belong to a prescribed set of measure close to 1. Here appears a remarkable new idea. Eskin and Mirzakhani observe that the time τ is essentially bilipschitz in ℓ (the precise statements on this essential point are given in [[11], Sect 7]). Therefore, we ask that τ avoids a set of time that has density close to 0 is equivalent to asking that ℓ avoids a set of time with density close to 0.

In [[11], Sect 11] Eskin and Mirzakhani introduce, for each (i,j), A-equivariant equivalence relation $\mathcal{C}_{ij}$ on leaves of W^+, whose atoms are, in a certain sense, the tangent distributions of E_{ij}. The fact that $q'_2 - q_2$ gets closer to a certain $E_{ij,\mathrm{bdd}}(q_2)$ guarantees that, when passing to the limit, we will be able to construct distinct points $\tilde{q}_2$ and $\tilde{q}'_2$, which belong to the same atom of $\mathcal{C}_{ij}$.

We want to use this construction to infer that the conditional measure $\nu_{W^+}(\tilde{q}_2)$ along $\mathcal{C}_{ij}$ has invariance under elements of $\mathcal{G}_+(\tilde{q}_2)$. We are going to construct points q_3 and q'_3 that rely dynamically on q_2 and q'_2 and that are very close to each other. For this we launch the dynamics of A again from q_2 and q'_2 for time t: we arrive then at points $q_3 = a_t q_1$ and $q'_3 = a_t q'_1$ which are very close, since $q'_1 \in W^-[q_1]$.

In [[11], Prop 11.4] Eskin and Mirzakhani show that by choosing the parameters correctly in this construction, we can deduce an invariance property of the conditional measures along $\mathcal{C}_{ij}$. Let's try to present this idea.

We want to compare the conditional measures of $\mathcal{C}_{ij}$ in q'_2, q'_2, q'_3 and q'_3. For this transport these measures along the tangent space. For (i,j) given, for ν-almost every x in X, let's write $\nu_{ij}(x)$ for the measure on $W^+(x)$ which is the inverse image of the conditional measure $\nu_{W^+}(x)$ along $\mathcal{C}_{ij}(x)$ via the map $W^+(x) \to W^+[x], v \mapsto x + v$.

Then, the link between the partition $\mathcal{C}_{ij}$ and the distribution $E_{ij,\mathrm{bdd}}$ (which we do not have explicit) implies that the measure $\nu_{ij}(q_2)$ is the image of $\nu_{ij}(uq_1)$ by a linear map which is essentially a similarity of ratio $\exp(\lambda_{ij}(uq_1, q_2))$. Similarly, $\nu_{ij}(q_3)$ is the image of $\nu_{ij}(q_1)$ by the linear map that is essentially a similarity with ratio $\exp(\lambda_{ij}(q_1, q_3))$. If u is not too big, we can guarantee that $\nu_{ij}(uq_1)$ and $\nu_{ij}(q_1)$ are close (it suffices to ask that q_1 and uq_1 belong to a set of measure close to 1 on which the map ν_{ij} is continuous). As one can also assume that $\lambda_{ij}(q_1, uq_1)$ is uniformly bounded, we deduce that if t is such that $\lambda_{ij}(q_2, q_3)$ is bounded, the measure $\nu_{ij}(q_3)$ is the image under the measure $\nu_{ij}(q_3)$[26] whose norm and the norm its inverse are uniformly bounded. In this case, as q_3 and q'_3 are close, if we can ensure that they belong to a set of continuity of ν_{ij}, $\nu_{ij}(q_3)$ is close to $\nu_{ij}(q'_3)$ and thus, by going to the limit $\nu_{ij}(\tilde{q}'_2)$ is the image of $\nu_{ij}(\tilde{q}_2)$ by a linear map. Now by construction, $\nu_{ij}(\tilde{q}_2)$ is the image under $\nu_{ij}(\tilde{q}_2)$[27] by an affine map that does not fix 0, since these two measures are inverse images in $W^+(\tilde{q}_2)$ and $W^+(\tilde{q}'_2)$ of the same measure on $W^+[\tilde{q}_2] = W^+[\tilde{q}'_2]$.

It follows that $\nu_{ij}(\tilde{q}_2)$ is invariant under a non-trivial affine-transformation, and therefore that the conditional measure along $\mathcal{C}_{ij}(\tilde{q}_2)$ is invariant under non-trivial affine transformation (which sends $\tilde{q}_2$ to $\tilde{q}'_2$). Note that, in this proof, all conditional measure are defined up to multiplication by a constant, but once invariance (up to a constant) is established, standard arguments allow to remove this difficulty.

[26] typo
[27] typo

Let's go back to the assumption we made: we asked that $\lambda_{ij}(q_2, q_3)$ and $\lambda_{ij}(q_2, q_3)$[28] is uniformly bounded and that q_3 and q_3' are contained in the same set of measure close to 1. The authors obtain the first properties by making sure that $\lambda_{ij}(uq_1, q_2) \approx \lambda_{ij}(u'q_1', q_2')$ and $\lambda_{ij}(q_1, q_3) \approx \lambda_{ij}(q_1', q_3')$. If that is the case, we fix now the value t by demanding that

$$\lambda_{ij}(q_2, q_3) = 0.$$

Then, on the one hand, $\lambda_{ij}(q_2, q_3)$ is bounded and, one the other hand, t becomes a function that is essentially bilipschitz in τ. Since τ depends on the same way of ℓ, τ is essentially a bilipschitz function of ℓ. Demanding that q_3 and q_3' avoid a set of small measure then becomes to demand that ℓ avoids a set of numbers of low density, which we shall do.

We still have to give some indication of how the authors make sure that $\lambda_{ij}(uq_1, q_2) \approx \lambda_{ij}(u'q_1', q_2')$ and $\lambda_{ij}(q_1, q_3) \approx \lambda_{ij}(q_1', q_3')$. This is one of the the big difficulties of the method. In the study of dynamical systems on homogeneous spaces, of which Eskin and Mirzakhani found their inspiration, this difficulty is completely absent, because the cocycles do not depend on the base point. In the case of strata, the authors solve this problem by using only linear cocycles that intervene group actions on flat bundles and therefore, in affine charts, these cocycles do not depend on the base point. These methods are employed in the proof of [[11], Prop 4.4] (which takes up ideas from Ledrappier [19]) which permit to control the behaviour of the Zimmer's reduction in the interior of a Lyapunov subspace when we change the base point. In [[11], Sect 11], one established by related methods results that allow to control the components of the Lyapunov decomposition. For example, let's mention the

Lemma 6.8 ([11] Lem 11.12)**.** *Let M be a differentiable manifold, (g_t) a flow on M and E a (g_t)-equivariant vector bundle. Suppose M has a g_t-invariant and ergodic Borel probability measure and let $x \mapsto \mathcal{V}_1(x) \oplus \cdots \oplus \mathcal{V}_r(x)$ the Osseledets decomposition of the action of g_t in E, associated with the Lyapunov exponents $\lambda_1 > \cdots > \lambda_r$. Then there exists $\alpha, \varepsilon > 0$ having the following property: for all $\delta > 0$ there exists $C > 0$ and a measurable set $Y \subset X$ of measure $\geq 1 - \delta$ such that, for all x and y in Y, for all $s > 0$, if $d(g_t x, g_t y) \leq \varepsilon$ for $|t| \leq s$, then, for all $1 \leq i \leq r$ we have*

$$d(\mathbb{P}(\mathcal{V}_i(x)), \mathbb{P}(\mathcal{V}_i(y))) \leq Ce^{-\alpha s}.$$

Proof. We choose Y so that there exists $\theta > 0$ and $0 < \rho < \frac{1}{2}\min_{1\leq i\leq r-1}(\lambda_i - \lambda_{i+1})$ such that, for all x in Y, for all real t, for all $1 \leq i \leq r$, for all v in $\mathcal{V}_i(x)$, one has

$$\theta \exp((\lambda_i - \rho)t)\|v\| \leq \|g_t v\| \leq \theta^{-1} \exp((\lambda_i + \rho)t)$$

and that, in addition, for all $1 \leq i < j \leq r$ the projective subspaces $\mathbb{P}(\mathcal{V}_i(x))$ and $\mathbb{P}(\mathcal{V}_j(x))$ remain at distance $\geq \theta$ from each other.

If x and y are as in the statement, the flat structure of the bundle and the fact that the orbits of x and y remain close to identify the fibre E_y with E_x (resp. $E_{g_s y}$ with $E_{g_s x}$, resp. $E_{g_{-s} y}$ with $E_{g_{-s} x}$) so that the action in these spaces under g_s (resp. g_{-s}) reads as the same linear map $A_s : E_x \to E_{g_s x}$ (resp. $A_{-s} : E_x \to E_{g_{-s} x}$[29]). Then, in $\mathcal{V}_1(x)$, A_s multiplies the norm by a factor of at least the order of $\exp((\lambda_1 - \rho)s)$ and, in $\mathcal{V}_2(x) \oplus \cdots \oplus \mathcal{V}_r(x)$, as in $\mathcal{V}_2(y) \oplus \cdots \oplus \mathcal{V}_r(y)$ the norm of A_s is dominated by $\exp((\lambda_2 + \rho)s) \ll \exp((\lambda_1 - \rho)s)$. By an elementary reasoning in linear algebra, we deduce that the projective spaces

$$\mathbb{P}(\mathcal{V}_2(x) \oplus \cdots \oplus \mathcal{V}_r(x)) \text{ and } \mathbb{P}(\mathcal{V}_2(y) \oplus \cdots \oplus \mathcal{V}_r(y))$$

[28] typo

[29] typo

are at distance $\ll \exp(-\alpha s)$ for some $\alpha > 0$. Letting α decrease, reasoning in the same way with A_{-s} we obtain

$$d(\mathbb{P}(\mathcal{V}_1(x)), \mathbb{P}(\mathcal{V}_1(y))) \ll e^{-\alpha s}.$$

The general case is obtained in a similar manner. □

To end this discussion of the drift argument, let us mention that we have given an incomplete version: indeed, nothing guarantees in our arguments that the affine transformation that preserves the measure $\mathcal{C}_{ij}(\tilde{q}_2)$ does not belong to the group $U^+(\tilde{q}_2)$. To get this extra property, we need to use an assumption from Proposition 6.7, namely that $\mathcal{L}^+(x)$ is not contained in $U^+(x)$.

If $U^+(x)$ is a vector subspace of $W^+(x)$, we take up the construction and ensure that in the proof, it is not the distance between q_2' and q_2 which is macroscopic, but the distance between q_2' and the affine subspace $q_2 + U^+(q_2) \subset W^+[q_2]$. Some work is needed, since the distribution $U^+ \subset W^+$ does not necessarily have an A-equivariant complement. It is necessary to choose a transversal and take into account possible deviations from this transversal.

In the general case, one simply has a distribution of Lie subalgebras $\mathfrak{u}^+$ of the distribution $\mathfrak{g}_+$ of Lie algebras of $\mathcal{G}_+(x)$, and we will have to guarantee that q_2' is located at distance bounded from below from $U^+(q_2)q_2 \subset W^+[q_2]$. In all of the proof, one replaces the estimates on the Lyapunov exponents, the Zimmer's reduction, etc., of the bundle W^+ by analogous objects in the complement of $\mathfrak{u}^+$ in $\mathfrak{g}^+$. A large number of difficulties posed by this extension arise and are dealt with by the results of [[11], Sect 6].

6.6. **The entropy argument.** We briefly mention here how Proposition 6.5 makes it possible to conclude the proof of Theorem 6.4. According to this proposition, we have AN-equivariant distributions of vector subspaces $\mathcal{L}^- \subset W^-$ and $\mathcal{L}^+ \subset W^+$ such that, for ν-almost every x, $\nu_{W^-}(x)$ is supported on $\mathcal{L}^-(x) + x \subset W^-[x]$ and $\nu_{W^+}(x)$ is $\mathcal{L}^+(x)$-invariant. As in the beginning of paragraph 6.4, we write $\mathcal{L}_\perp \subset H^1_\perp$ for the A-equivariant distribution for which $\mathcal{L}^- \cap \mathbb{R}e_2 \otimes H^1_\perp = \mathbb{R}e_2 \otimes \mathcal{L}_\perp$. Finally, we denote, for ν-almost every x, $U(x) \supset \mathcal{L}_\perp(x)$ the vector subspace of $H^1_\perp$ such that $\mathbb{R}e_1 \otimes e_1 \oplus \mathbb{R}e_1 \otimes U(x)$ is the connected component of the stabiliser of $\nu_{W^+}(x)$ in $W^+(x)$ ($\mathbb{R}e_1 \otimes e_1$ stabilises $\nu_{W^+}(x)$ because it is the tangent direction of N and N preserves ν).

We let $(\lambda_i)_{i\in I}$ denote the Lyapunov exponents of the Kontsevich-Zorich cocycle in U counted with multiplicities, and let $J \subset I$ be such that the Lyapunov exponents in $\mathcal{L}^+$ are $(\lambda_i)_{i\in J}$. According to [[11], Thm A.3] (which is, inspired by ideas of Forni [13], taken from Forni, Matheus, Zorich [14]), as the distribution U is AN-equivariant, we have

$$\sum_{i\in I} \lambda_i \geq 0.$$

We can estimate the entropy of A for the measure ν in terms of the conditional measures of ν along the stable and unstable foliations. This estimates come from the Ledrappier-Young entropy formula [20], [21] used for rigidity problems by Margulis and Tomanov [23] and adapted here by Eskin and Mirzakhani [[11], Thm B.7]. The essence of this formula is that the entropy is the sum of the positive (or negative) Lyapunov exponents multiplied by the dimension of the measure along the foliation associated to the exponents.

In this context, this formula gives us, since for ν-almost every x in X, $\nu_{W^-}(x)$ is supported on $\mathcal{L}^-(x) + x$,

$$h(a_1, \nu) \leq 2 + \sum_{i\in J}(1 - \lambda_i) \leq 2 + \sum_{i\in I}(1 - \lambda_i) \leq 2 + |I|$$

(where we used that the λ_i are ≤ 1 and that $\sum_{i\in I}\lambda_i \geq 0$). But, since for ν-almost every x, $\nu_{W^+}(x)$ is $U(x)$-invariant and N-invariant, the same formula implies that

$$h(a_1,\nu) \geq 2 + \sum_{i\in I}(1+\lambda_i) \geq 2 + |I|.$$

All inequalities are therefore equalities. In particular, we have $\mathcal{L}_\perp = U$, $\mathbb{R}e_2 \otimes e_2 \subset \mathcal{L}^-$ and the case of equality in the Ledrappier-Young formula implies that for ν-almost every x in X, $\nu_{W^-}(x)$ is invariant under translations by $\mathcal{L}^-$. The measure ν is therefore N_--invariant (where N_- is the group of lower triangular unipotent matrices in $\mathrm{SL}_2(\mathbb{R})$), and since N and N_- generate $\mathrm{SL}_2(\mathbb{R})$, ν is $\mathrm{SL}_2(\mathbb{R})$-invariant. Similarly, the distribution $\mathcal{L}_\perp$[30] is both N_- and N-equivariant: it is therefore $\mathrm{SL}_2(\mathbb{R})$-equivariant.

6.7. **Random Walks.** Let us briefly mention the end of the proof of Theorem 3.4 once Theorem 6.4 is established. These arguments are developed in [[11], Sect 14-16]. We have a P-invariant and ergodic probability measure ν on $X = \mathcal{H}_1(\alpha)$. We are trying to show that X is affine. According to Theorem 6.4, we know that ν is $\mathrm{SL}_2(\mathbb{R})$-invariant and there is a measurable distribution $\mathcal{L}$ of subspaces of $T\mathcal{H}(\alpha)$, defined ν-almost everywhere, which is $\mathrm{SL}_2(\mathbb{R})$-equivariant, and such that, for ν-almost all x, the conditional measure $\nu_{W^+}(x)$ is the Lebesgue measure of $x + (\mathcal{L}(x) \cap W^+(x))$. The distribution $\mathcal{L}$ is the candidate to be the tangent distribution of the affine subspace that is supported by ν.

To show that ν is affine, it suffices to show that there exists a set of x of measure > 0 for which there exists an open set Ω of $\mathcal{L}(x)$ with $\nu(\Omega + x) > 0$. Assuming the contrary, Eskin and Mirzakhani manage to build, for any set of positive measure, elements x and y with $y \in W^+[x]$ but $y \notin x + (W^+(x) \cap \mathcal{L}(x))$ which is a contradiction.

This construction is based on a drifting argument similar to the previous one, but for a different dynamical system.

Let's construct this dynamical system. We fix a probability measure μ on $\mathrm{SL}_2(\mathbb{R})$ that is left- and right-$\mathrm{SO}(2)$-invariant which is of compact support and in the Lebesgue measure class. We denote by B_+ (resp. B_-) the space of sequences $(b_0, b_1, \dots)$ (resp. $(\dots, b_{-2}, b_{-1})$) of elements in $\mathrm{SL}_2(\mathbb{R})$, which is provided with the product measure $\beta_+ = \mu^{\otimes \mathbb{N}}$ (resp. $\beta_- = \mu^{\otimes \mathbb{Z}_-^*}$). We denote by $B = B_- \times B_+$ the space of sequences indexed by $\mathbb{Z}$, which is also provided with the product measure $\mu^{\otimes \mathbb{Z}}$. Finally, we denote $T : B \to B$ the shift transformation.

Note that if $b = (b_-, b_+)$ is an element of B, we can consider all the sequences of the form (b_-, a) with a in B_+ as local unstable leaf of b for the action of T on B. The idea of the second stage of the proof consists of playing the game on this local unstable leaf with this role being played by U^+ in the first stage.

More precisely, let's introduce the dynamical system

$$T^X : B \times X \to B \times X, (b,x) \mapsto (Tb, b_0 x).$$

The measurable transformation preserves the product measure $\beta \otimes \nu$, which is ergodic for general reasons.

The second stage of the proof is then established by a process analogous to that, which makes it possible to obtain the first, by replacing the transformations $q_1 \mapsto uq_1$ and $q_1' \mapsto u'q_1'$ by the transformations of the type $q_1 = (b_1, x_1) \mapsto ((b_{1,-}, a), x)$ and $q_1 = (b_1, x_1') \mapsto ((b_{1,-}, a), x')$ with a in B_+.

[30] typo

Some of the difficulties that arose during the first space (like having to introduce inert subspaces, bounded subspaces, etc.) do not show up because the Kontsevich-Zorich cocycle has better ergodic properties over $\mathrm{SL}_2(\mathbb{R})$-invariant measures then over measures that are A-invariant. The proof uses in particular the

Theorem 6.9 ([11], Thm A.6)**.** *Let ν be a $\mathrm{SL}_2(\mathbb{R})$-invariant ergodic probability measure on $X = \mathcal{H}_1(\alpha)$. Then the Zariski closure of the KZ cocycle in $H^1(S, \mathbb{R})$ above of ν is a semi-simple group.*

It is not clear that the Zariski closure of the cocycle on $H^1(S, \Sigma, \mathbb{R}^2)$ is semi-simple. To overcome the problems posed by this difficulty, the authors use a result on the geometry of the invariant measures. Let $\mathcal{F} \subset H^1(S, \mathbb{R})$ the sum of $\mathrm{SL}_2(\mathbb{R})$-equivariant distributions defined ν-almost everywhere where all Lyapunov exponents of A are zero.

Theorem 6.10 ([2])**.** *Let ν be an $\mathrm{SL}_2(\mathbb{R})$-invariant and ergodic probability measure on $X = \mathcal{H}^1(\alpha)$. Then there exists $Y \subset X$ of measure 1 for ν such that for all x in Y, and for all y in Y sufficiently close to x, we have $p(y - x) \in \mathbb{R}^2 \oplus \mathcal{F}(x)^{\perp}$.*

In this formula, $\mathcal{F}(x)^{\perp}$ means the orthogonal of $\mathcal{F}(x)$ for the symplectic form $\overline{\omega}$.

References

[1] J. S. Athreya, *Quantitative recurrence and large deviations for Teichmuller geodesic flow*, Geom. Dedicata, 119 (2006), pp. 121–140.

[2] A. Avila, A. Eskin, and M. Möller, *Symplectic and isometric sl (2, r) invariant subbundles of the hodge bundle*, arXiv preprint arXiv:1209.2854, (2012).

[3] R. Azencott, *Espaces de Poisson des groupes localement compacts*, Lecture Notes in Mathematics, Vol. 148, Springer-Verlag, Berlin-New York, 1970.

[4] Y. Benoist and J.-F. Quint, *Mesures stationnaires et fermés invariants des espaces homogènes*, Ann. of Math. (2), 174 (2011), pp. 1111–1162.

[5] P. Bougerol and J. Lacroix, *Products of random matrices with applications to Schrödinger operators*, vol. 8 of Progress in Probability and Statistics, Birkhäuser Boston, Inc., Boston, MA, 1985.

[6] S. G. Dani and G. A. Margulis, *Asymptotic behaviour of trajectories of unipotent flows on homogeneous spaces*, Proc. Indian Acad. Sci. Math. Sci., 101 (1991), pp. 1–17.

[7] M. Einsiedler and A. Katok, *Invariant measures on G/Γ for split simple Lie groups G*, Comm. Pure Appl. Math., 56 (2003), pp. 1184–1221. Dedicated to the memory of Jürgen K. Moser.

[8] M. Einsiedler, A. Katok, and E. Lindenstrauss, *Invariant measures and the set of exceptions to Littlewood's conjecture*, Ann. of Math. (2), 164 (2006), pp. 513–560.

[9] A. Eskin, G. Margulis, and S. Mozes, *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*, Ann. of Math. (2), 147 (1998), pp. 93–141.

[10] A. Eskin and H. Masur, *Asymptotic formulas on flat surfaces*, Ergodic Theory Dynam. Systems, 21 (2001), pp. 443–478.

[11] A. Eskin and M. Mirzakhani, *Invariant and stationary measures for the sl (2, r) action on moduli space*, arXiv preprint arXiv:1302.3320, (2013).

[12] A. Eskin, M. Mirzakhani, and A. Mohammadi, *Isolation, equidistribution, and orbit closures for the* $\mathrm{SL}(2, \mathbb{R})$ *action on moduli space*, Ann. of Math. (2), 182 (2015), pp. 673–721.

[13] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Ann. of Math. (2), 155 (2002), pp. 1–103.

[14] G. Forni, C. Matheus, and A. Zorich, *Lyapunov spectrum of invariant subbundles of the Hodge bundle*, Ergodic Theory Dynam. Systems, 34 (2014), pp. 353–408.

[15] H. Furstenberg, *Noncommuting random products*, Trans. Amer. Math. Soc., 108 (1963), pp. 377–428.

[16] ———, *A Poisson formula for semi-simple Lie groups*, Ann. of Math. (2), 77 (1963), pp. 335–386.

[17] A. Katok and R. J. Spatzier, *Invariant measures for higher-rank hyperbolic abelian actions*, Ergodic Theory Dynam. Systems, 16 (1996), pp. 751–778.

[18] M. Kontsevich, *Lyapunov exponents and Hodge theory*, in The mathematical beauty of physics (Saclay, 1996), vol. 24 of Adv. Ser. Math. Phys., World Sci. Publ., River Edge, NJ, 1997, pp. 318–332.

[19] F. Ledrappier, *Positivity of the exponent for stationary sequences of matrices*, in Lyapunov exponents (Bremen, 1984), vol. 1186 of Lecture Notes in Math., Springer, Berlin, 1986, pp. 56–73.

[20] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula*, Ann. of Math. (2), 122 (1985), pp. 509–539.

[21] ———, *The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension*, Ann. of Math. (2), 122 (1985), pp. 540–574.

[22] E. Lindenstrauss, *Invariant measures and arithmetic quantum unique ergodicity*, Ann. of Math. (2), 163 (2006), pp. 165–219.

[23] G. A. Margulis and G. M. Tomanov, *Invariant measures for actions of unipotent groups over local fields on homogeneous spaces*, Invent. Math., 116 (1994), pp. 347–392.

[24] H. Masur, *Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential*, in Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986), vol. 10 of Math. Sci. Res. Inst. Publ., Springer, New York, 1988, pp. 215–228.

[25] ———, *The growth rate of trajectories of a quadratic differential*, Ergodic Theory Dynam. Systems, 10 (1990), pp. 151–176.

[26] C. T. McMullen, *Dynamics of* $SL_2(\mathbb{R})$ *over moduli space in genus two*, Ann. of Math. (2), 165 (2007), pp. 397–456.

[27] S. Meyn and R. L. Tweedie, *Markov chains and stochastic stability*, Cambridge University Press, Cambridge, second ed., 2009. With a prologue by Peter W. Glynn.

[28] E. Nummelin, *General irreducible Markov chains and nonnegative operators*, vol. 83 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1984.

[29] M. Ratner, *On Raghunathan's measure conjecture*, Ann. of Math. (2), 134 (1991), pp. 545–607.

[30] ———, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke Math. J., 63 (1991), pp. 235–280.

[31] M. Seppälä and T. Sorvali, *Geometry of Riemann surfaces and Teichmüller spaces*, vol. 169 of North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, 1992.

[32] J. M. Steele, *Kingman's subadditive ergodic theorem*, Ann. Inst. H. Poincaré Probab. Statist., 25 (1989), pp. 93–98.

[33] W. A. Veech, *Moduli spaces of quadratic differentials*, J. Analyse Math., 55 (1990), pp. 117–171.

[34] J.-C. Yoccoz, *Interval exchange maps and translation surfaces*, in Homogeneous flows, moduli spaces and arithmetic, vol. 10 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2010, pp. 1–69.

[35] R. J. Zimmer, *Ergodic theory and semisimple groups*, vol. 81 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1984.