Cut-and-Project Quasicrystals: Patch Frequency and Moduli Spaces

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Definition: Delone set

Let (X, d) be a metric space. A point set $\Lambda \subset X$ is called **Delone** if

- Λ is uniformly discrete
  $\exists r > 0$ such that $B_r(x) \cap \Lambda = \{x\}$ for all $x \in \Lambda$
- Λ is relatively dense $\exists R > 0$ such that $B_R(0) + \Lambda$ covers X.

A local criterion for regularity of a system of points.

Definition: T-patch

A point set $\mathcal{P}(x, T) = \Lambda \cap B_T(x)$ for $x \in \Lambda$ is called T-patch.

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Thm(Delone,Dolbilin,Shtogrin,Galiulin;'76)

Let $\Lambda \subset \mathbb{R}^d$ be Delone. There exists $T_0 > 0$ such that if all T_0-patches are equivalent up to isometry then there exists a lattice $\Gamma < \mathrm{Isom}(\mathbb{R}^d)$ and $x \in \Lambda$ s.t.

$$\Lambda = \Gamma x.$$

Same paper: $\mathbb{S}^n$, $\mathbb{H}^n$.

Meyer's definition of a quasicrystal

A Delone set $\Lambda \subset \mathbb{R}^d$ is called Meyer if $\Lambda - \Lambda \subset \Lambda + E$ for $E \subset \mathbb{R}^d$ finite.

This generalizes lattices:

- If E is trivial, Λ is a lattice.
- Λ is Meyer if and only if Λ and $\Lambda - \Lambda$ are Delone. (Lagarias)

Embedding Theorem (Meyer '72)

Any Meyer set is a subset of a cut-and-project quasicrystal.

Definition: cut-and-project scheme

The data $(\mathcal{L}, \mathbb{R}^d, \mathbb{R}^m)$ defines a cut-and-project scheme if

$\mathcal{L}$ is a lattice in the group $\mathbb{R}^d \times \mathbb{R}^m$

$\pi = \pi_{\mathbb{R}^d}, \pi_{\text{int}} = \pi_{\mathbb{R}^m}$ natural projections satisfy

(I) $\pi|_{\mathcal{L}}$ is injective

(D) $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathbb{R}^m$.

Definition: Window and cut-and-project sets

Window $\mathcal{W} \subset \mathbb{R}^m$ bounded and define the cut-and-project set

$$\Lambda(\mathcal{W}, \mathcal{L}) = \pi\left(\mathcal{L} \cap \left(\mathbb{R}^d \times \mathcal{W}\right)\right)$$

If $\mathcal{W}$ has non-empty interior, call it cut-and-project quasicrystal. It is Meyer.

(Regular) boundary of $\mathcal{W}$ has zero measure

For any T-patch $\mathcal{P} = \mathcal{P}(y, T) = \Lambda \cap B_T(y)$ define

$$\mathrm{freq}_\Lambda(\mathcal{P}) = \lim_{t\to\infty} \frac{\#\{x \in \Lambda \cap B_t(0) : \mathcal{P}(x, T) \sim_{\mathbb{R}^d} \mathcal{P}\}}{t^d}.$$

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Theorem on Patch Frequency Asymptotics

Fix $d + m = n$, $d > 2$. There exists $\kappa > 0$ such that for a **random** cut-and-project quasicrystal Λ of $(\mathcal{L}, \mathbb{R}^d, \mathbb{R}^m)$ of $\mathbb{K}$-type SL_k or Sp_k we have

$$\#\{x \in \Lambda \cap B_t(0) : \mathcal{P}(x, T) \sim_{\mathbb{R}^d} \mathcal{P}\} = \mathrm{freq}_\Lambda(\mathcal{P}) t^d + \mathcal{O}(t^{d-\kappa}).$$

Construction of Marklof and Strömbergsson

- $\Lambda = \Lambda(\mathcal{L}, \mathcal{W})$ cut-and-project quasicrystal of $(\mathcal{L}, \mathbb{R}^d, \mathbb{R}^m)$
- Suppose $\mathcal{L} = g\mathbb{Z}^n \in X_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ for $g \in \mathrm{SL}_n(\mathbb{R})$ and consider $\mathrm{SL}_d(\mathbb{R}) < \mathrm{SL}_n(\mathbb{R})$ top-left block.
- $\overline{\mathrm{SL}_d(\mathbb{R})\mathcal{L}} = L'\mathcal{L}$ for some $L' < \mathrm{SL}_n(\mathbb{R})$ (Ratner)

The space of quasicrystals associated to $\mathcal{L}$

$$\mathfrak{Q} = \{\Lambda(y, \mathcal{W}) : y \in L'\mathcal{L}\}$$

with probability measure $\mu_{\mathfrak{Q}}$ (push forward of $m_{L'\mathcal{L}}$ under $y \mapsto \Lambda(y, \mathcal{W})$).

MS actually consider $\mathrm{ASL}_d(\mathbb{R})$-orbit closure. See also El-Baz for Adelic case.

For $m_{L'\mathcal{L}}$-a.e. y defines a cut-and-project scheme i.e. it holds

(I) $\pi|_{\mathcal{L}}$ is injective

(D) $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathbb{R}^m$.

The space of quasicrystals associated to $\mathcal{L}$

$$\mathfrak{Q} = \{\Lambda(y, \mathcal{W}) : y \in L'\mathcal{L}\} \subset \mathfrak{C}(\mathbb{R}^d)$$

where $\mathfrak{C}(\mathbb{R}^d)$ is the space of closed sets on $\mathbb{R}^d$ equipped with the Chaubauty-Fell metric,

$$d(\Lambda_0, \Lambda_1) = \max\left(1, \inf_{\varepsilon}\left(\Lambda_i \cap B_{\frac{1}{\varepsilon}}(0) \subset \Lambda_{1-i} + B_{\varepsilon}(0);\ i = 0, 1\right)\right)$$

Proposition

$$\Psi : X_n = \mathrm{SL}_n(\mathbb{R})/\,\mathrm{SL}_n(\mathbb{Z}) \to \mathfrak{C}(\mathbb{R}^d), \quad \mathcal{L} \mapsto \Lambda(\mathcal{L}, \mathcal{W})$$

is Borel, and continuous when $\pi_{\mathrm{int}}(\mathcal{L}) \cap \partial\mathcal{W} = \emptyset$.

The space of quasicrystals associated to $\mathcal{L}$

$$\mathfrak{Q} = \{\Lambda(y, \mathcal{W}) : y \in L'\mathcal{L} = \overline{\mathrm{SL}_d(\mathbb{R})\mathcal{L}}\} \subset \mathfrak{C}(\mathbb{R}^d)$$

Proposition

$$\Psi : X_n = \mathrm{SL}_n(\mathbb{R})/\,\mathrm{SL}_n(\mathbb{Z}) \to \mathfrak{C}(\mathbb{R}^d), \quad \mathcal{L} \mapsto \Lambda(\mathcal{L}, \mathcal{W})$$

is Borel, and continuous when $\pi_{\mathrm{int}}(\mathcal{L}) \cap \partial\mathcal{W} = \emptyset$.

Siegel-Veech Theorem of MS14

If $\mathcal{W}$ is regular then $m_{L'\mathcal{L}}(\pi_{\mathrm{int}}(\mathcal{L}) \cap \partial\mathcal{W} \neq \emptyset) = 0$

Corollary

$\mu_{\mathfrak{Q}} = \Psi_* m_{L'\mathcal{L}}$ is a Borel probability and Ψ_* is continuous at $m_{L'\mathcal{L}}$.

Solidarity Theorem

Suppose μ on $\mathfrak{C}(\mathbb{R}^d)$ is $\mathsf{SL}_d(\mathbb{R})$-ergodic probability gives positive mass on cut and project sets with regular windows.
Then $\mu = \mu_{\mathfrak{Q}}$ for some $\mathfrak{Q}$.
In particular, there is a single internal space and window that give rise to its support.

Proof.

By the Howe-Moore Theorem, μ is g_t ergodic (g_t one-parameter diagonalizable subgroup)
By the Birkhoff Ergodic Theorem, for μ a.e. Λ

$$\frac{1}{T}\int_0^T (g_t)_* \delta_\Lambda dt \to \mu$$

□

Proof.

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By the Birkhoff Ergodic Theorem, for μ a.e. Λ

$$\frac{1}{T}\int_0^T (g_t)_*\delta_\Lambda dt \to \mu$$

By invariance of $U = G^+_{g_t}$ and Fubini's theorem, a.e. Λ, m_U-a.e. $u \in U$, $u.\Lambda$ is Birkhoff generic.

Hence for $\Omega \subset U$ compact, non-empty interior

$$\frac{1}{T}\int_0^T \int_\Omega (g_t u)_*\delta_\Lambda dm_U dt \to \mu$$

□

Proof.

For μ-a.e. Λ

$$\frac{1}{T}\int_0^T \int_\Omega (g_t u)_* \delta_\Lambda dm_U dt \to \mu$$

Take such a generic $\Lambda = \Lambda(\mathcal{L}_\Lambda, W) = c\Lambda(\mathcal{L}, \frac{1}{c}W)$ for some unimodular $\mathcal{L} < \mathbb{R}^n$ and regular window $W < \mathbb{R}^m$.
Consider the same ergodic average, but on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ replacing Λ with $\mathcal{L}$.
It converges to some homogeneous measure $m_{L'\mathcal{L}}$ by a theorem of Shah!
Since any such $m_{L'\mathcal{L}}$ is a continuity point for Ψ_* by previous corollary, also the original ergodic average converges to $\Psi_* m_{L'\mathcal{L}}$.

□

Theorem I

Write $Y = \overline{\mathsf{SL}_d(\mathbb{R})g\,\mathsf{SL}_n(\mathbb{Z})} = L'\mathcal{L} = gL\,\mathsf{SL}_n(\mathbb{Z})$. Then L is isogeneous to the real points of an almost $\mathbb{Q}$-simple linear algebraic group $\mathbb{L}$.

Theorem II

$\mathbb{L}$ is isogeneous (over $\mathbb{C}$) to either a product of $SL_k(\mathbb{C})$s or $\mathsf{Sp}_k(\mathbb{C})$s.

What we really need

What $\mathbb{Q}$-groups do really appear?
For the remainder of this talk: $\mathbb{L} = \mathsf{Res}_{\mathbb{K}|\mathbb{Q}}(\mathsf{SL}_d)$

(I) $\pi|_{\mathcal{L}}$ is injective

(D) $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathbb{R}^m$.

(I) and (D) implies (L) Lemma

A vector space $V < \mathbb{R}^n$ is called $\mathcal{L}$-rational if $V \cap \mathcal{L}$ is a lattice in V.

The following implications hold.

a. **(D)** $\Rightarrow \mathbb{R}^d$ is not contained in a proper $\mathcal{L}$-rational subspace.

b. **(I)** $\Rightarrow \mathbb{R}^m$ contains no non-trivial $\mathcal{L}$-rational subspace.

Define

(irred) There exists no proper $\mathcal{L}$-rational subspace of $\mathbb{R}^n$ that is $\mathrm{SL}_d(\mathbb{R})$-invariant

c. **(I)** and **(D)** $\Rightarrow$ **(irred)**.

Proof.

- Shah '91: $\mathbb{L}$ minimal $\mathbb{Q}$-group generated by unipotents containing $H' = g^{-1}\,\mathsf{SL}_d(\mathbb{R})g$
- $\mathbb{L}$ is semi-simple: Let $\mathbb{U}$ denote the (unipotent) radical of $\mathbb{L}$. By Shah it is also defined over $\mathbb{Q}$.
- $V^{\mathbb{U}}$ is a rational subspace by Zariski density of $\mathbb{U}(\mathbb{Q})$. Cannot be by previous lemma, so it must be trivial.
- Since $\mathbb{U}$ unipotent it does have an invariant subspace, so it $\mathbb{U}$ must be trivial.

$\square$

$\overline{\mathsf{SL}_d(\mathbb{R})g\,\mathsf{SL}_n(\mathbb{Z})} = L'\mathcal{L} = g\mathbb{L}(\mathbb{R})\,\mathsf{SL}_n(\mathbb{Z})$. Claim: $\mathbb{L}$ is $\mathbb{Q}$-simple

Proof.

- Decompose $\mathbb{L}$ into $\mathbb{Q}$-simple factors $\mathbb{T}_j$ which we further split in $\mathbb{R}$-simple factors $\mathbb{S}_i$.
- Need to show that $H' = g^{-1}\,\mathsf{SL}_d(\mathbb{R})g < S_i = \mathbb{S}_i(\mathbb{R})$ for some i.
- Decompose $\mathbb{R}^n$ into irreducible representations of H', i.e. $\mathbb{R}^n = g^{-1}\mathbb{R}^d \oplus g^{-1}\mathbb{R}^m$.
- Project H' to S_j, call it H'_j. $\mathcal{I} = \{i : H'_i \neq e\}$. Let $L_{H'} = \prod_{i\in\mathcal{I}} S_i$. Contains H'.
- Let V_i isotypical reps of $L_{H'}$, assume $g^{-1}\mathbb{R}^d < V_1$. Then H' is inside intersection of kernels of $L_{H'}|_{V_{>1}}$.
- Intersection of these kernels is a normal subgroup of $L_{H'}$ hence equal to some subproduct of S_i, $i \in \mathcal{I}$, contains H', so it is all of $L_{H'}$. Hence $V_{>1}$ trivial rep.

□

$\overline{\mathrm{SL}_d(\mathbb{R})g\,\mathrm{SL}_n(\mathbb{Z})} = L'\mathcal{L} = g\mathbb{L}(\mathbb{R})\,\mathrm{SL}_n(\mathbb{Z})$. **Claim:** $\mathbb{L}$ **is** $\mathbb{Q}$**-simple**

Proof.

- Need to show that $H' = g^{-1}\,\mathrm{SL}_d(\mathbb{R})g < S_i = \mathbb{S}_i(\mathbb{R})$ for some i. So far: $\mathbb{R}^n = V_1 \oplus W$ where W trivial rep of $L_{H'}$ and $g^{-1}\mathbb{R}^d < V_1$.
- $H'_i = \pi_{S_i}(H')$, H'_i also preserve $g^{-1}\mathbb{R}^d$ (argument using regular elements $h = \prod h_i$).
- Assume $h_1 \in H'_2$ action on $g^{-1}\mathbb{R}^d$ non-trivial, take some h_1-invariant proper subspace V'' of $g^{-1}\mathbb{R}^d$.
- Since $H'_{>1}$-invariant, must be trivial w.r.t $H'_{>1} \simeq \mathrm{SL}_d(\mathbb{R})$. Use here that: 1) simplicity 2) classification of reps of $\mathrm{SL}_d(\mathbb{R})$ 3) and that dim of V'' is $< d$.
- Get non trivial kernels in $S_{>1}$, so kernels equal to S_j and S_j act trivally on V'' and hence on $V''' = \mathrm{span}(S_1 V'')$.
- So V''' is $L_{H'}$-invariant. But note that since $H' < L_{H'}$, $L_{H'}$ acts irred on V_1, hence $V_1 = V'''$.

□

$\overline{\mathsf{SL}_d(\mathbb{R})g\,\mathsf{SL}_n(\mathbb{Z})} = L'\mathcal{L} = g\mathbb{L}(\mathbb{R})\,\mathsf{SL}_n(\mathbb{Z})$. Claim: $\mathbb{L}$ is $\mathbb{Q}$-simple

Proof.

- Hence $S_{>1}$ acts trivial on V_1, hence on $\mathbb{R}^n$ by previous slide. Hence $S_{>1}$ didn't exist in the first place (definition if $\mathcal{I}$), and $H' < S_1$.

$\square$

Theorem (Tamagawa)

If $\mathbb{L}$ is almost $\mathbb{Q}$-simple and simply connected, there exists a field $\mathbb{K}$ and an absolutely almost simple simply connected group $\mathbb{G}$ defined over $\mathbb{K}$ such that $\mathbb{L} \simeq_{\mathbb{Q}} Res_{\mathbb{K}/\mathbb{Q}}(\mathbb{G})$.

Theorem (Morris-Witte in Appendix of Solomon-Weiss '14)

If $\mathbb{G}$ as above contains a conjugate of the top left $\mathrm{SL}_d(\mathbb{C})$ then $\mathbb{G}$ is either of type A_k or C_k.

Let $\mathbb{K}$ be a number field.
Let $\mathbb{G}$ be a $\mathbb{K}$-group.
Then there is a $\mathbb{Q}$-group $\mathrm{Res}_{\mathbb{K}|\mathbb{Q}}(\mathbb{G})$ such that
if $\sigma_i : \mathbb{K} \to \mathbb{C}$ fields embeddings then

$$\mathrm{Res}_{\mathbb{K}|\mathbb{Q}}(\mathbb{G})(\mathbb{Q}) = \{\sigma_1(g), \ldots, \sigma_d(g) : g \in \mathbb{G}(\mathbb{K})\}$$

(similarly, with $\mathbb{Z}$ and $\mathcal{O}$ replacing $\mathbb{Q}$ resp. $\mathbb{K}$)
For any field $\mathbb{K}'$ containing all $\sigma_i(\mathbb{K})$,

$$\mathrm{Res}_{\mathbb{K}|\mathbb{Q}}(\mathbb{G})(\mathbb{K}') = \prod \mathbb{G}^{\sigma_i}(\mathbb{K}').$$

Concretely, $\mathbb{K} = \mathbb{Q}(\sqrt{2})$, $\mathbb{G} = \mathrm{SL}_2(\mathbb{C})$.
$\mathbb{K}$ is a $\mathbb{Q}$ vectorspace. In fact algebra.

$$\phi : a + b\sqrt{2} \mapsto \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \in \mathrm{Mat}_2(\mathbb{Q})$$

$$AD - BC = 1 \qquad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} \phi(A) & \phi(B) \\ \phi(C) & \phi(D) \end{bmatrix} \in \mathrm{SL}_4(\mathbb{C})$$

Image of ϕ can be simulatenously diagonalized (over $\mathbb{K}$), $\phi(z)$ has Eigenvalues $z = \sigma_1(z)$, $\overline{z} = \sigma_2(z)$.
Applying this diagonalization, and a permutation matrix,

$$\begin{bmatrix} \phi(A) & \phi(B) \\ \phi(C) & \phi(D) \end{bmatrix} \mapsto \begin{bmatrix} \sigma_1(A) & \sigma_1(B) & & \\ \sigma_1(C) & \sigma_1(D) & & \\ & & \sigma_2(A) & \sigma_2(B) \\ & & \sigma_2(C) & \sigma_2(D) \end{bmatrix} \in \mathrm{SL}_2^{\sigma_1}(\mathbb{C}) \times \mathrm{SL}_2^{\sigma_2}(\mathbb{C}$$

- $\mathbb{K}$ is a totally real number field of degree D. $\mathcal{O}$ ring of integers.
- Field embeddings $\sigma_i : \mathbb{K} \to \mathbb{R}$, extend to $\mathbb{K}^k \to \mathbb{R}^k$.
- $\mathcal{L} = \{(\sigma_1(v), \ldots, \sigma_D(v)) : v \in \mathcal{O}^k\} < \mathbb{R}^{Dk}$ lattice

Example: Vertices of Ammann-Beenker Tiling

$\mathbb{K} = \mathbb{Q}(\sqrt{2})$, $k = 2$ then consider
$\{(\sigma_1(v), \sigma_2(v)) : v \in \mathbb{Z}[\sqrt{2}]^2\} < \mathbb{R}^4$ with basis

$$\begin{bmatrix} 1 & 0 & \sqrt{2} & 0 \\ 0 & 1 & 0 & \sqrt{2} \\ 1 & 0 & -\sqrt{2} & 0 \\ 0 & 1 & 0 & -\sqrt{2} \end{bmatrix}$$

Ammann-Beenker: $\mathcal{L}$ finite index sublattice of above (s.t. $v_1 - v_2 \in \sqrt{2}\mathbb{Z}[\sqrt{2}]$).
$d = m = 2$
$\mathcal{W}$ =Octagon.

See Baake-Grimm or Hammarhjelm.

Demo

$f \in C_c(\mathbb{R}^d)$. Define the Siegel-Veech transform, for $\Lambda \in \mathfrak{Q}$,

$$\hat{f}(\Lambda) = \sum_{v \in \Lambda - \{0\}} f(v)$$

Siegel-Veech formula, MS 2014

$$\mu_{\mathfrak{Q}}(\hat{f}) = c m_{\mathbb{R}^d}(f)$$

Reduces to Siegel formula on (sub-)space of lattices, applied to $f \times \mathbb{1}_{\mathcal{W}}$

- $\mathbb{K}$ totally real numberfield, $\mathcal{O}$ its ring of integers and $\sigma_i : \mathbb{K} \to \mathbb{R}$ embeddings, $i = 1 \dots D = \deg(\mathbb{K})$
- $\mathbb{K}_S = \prod \mathbb{K}_{|\cdot|_{\sigma_i}} = \prod \mathbb{R} = \mathbb{R}^D$
- $\mathcal{O}^d = \mathcal{L}_{\mathcal{O}} < \mathbb{K}_S^d$ lattice embedded via $v \mapsto (\sigma_1(v), \dots, \sigma_D(v))$
- $G = \mathrm{SL}_d(\mathbb{K}_S) = \mathrm{SL}_d(\mathbb{R})^D$, $\Gamma = \Gamma_{\mathcal{O}} = \mathrm{SL}_d(\mathcal{O}) < G$, embedded via $\gamma \mapsto (\sigma_1(\gamma), \dots, \gamma_D(\gamma))$.

Goal: Roger type bounds

Suppose $f \in C_c(\mathbb{R}^d)$, $\Lambda \in \mathfrak{Q} = \Psi(G/\Gamma) = \{\Lambda(\mathcal{L}, \mathcal{W}) : \mathcal{L} \in G.\mathcal{O}^d\}$, $\mu_{\mathfrak{Q}} = \Psi_* m_{G/\Gamma}$ then $\mu_{\mathfrak{Q}}(\widehat{f}^2) = m_{\mathbb{R}^d}(f)^2 + \mathcal{O}(m_{\mathbb{R}^d}(f))$.

What I will prove: If $f = f_1 \cdots \times f_D : \mathbb{R}^n \to \mathbb{R}$ characteristic function and $\hat{f}$ Siegel transform on $\mathbb{R}^n$ as usual then

$$m_{G/\Gamma}(\widehat{f}^2) = m_{\mathbb{R}^n}(f)^2 + \mathcal{O}(m_{\mathbb{R}^n}(f)).$$

Integrability

$$\hat{f} \in L^{d+\varepsilon}(m_{G/\Gamma}).$$

This is strict.

Eskin-Margulis-Mozes '99 for $\mathbb{K} = \mathbb{Q}$.

Integrability

If $f \in C_c(\mathbb{R}^{dD=n})$, then

$$\hat{f} \in L^{d+\varepsilon}(m_{\mathrm{SL}_d(\mathbb{K}_S)/\Gamma_{\mathcal{O}}})$$

where $\mathrm{SL}_d(\mathbb{K}_S) = \mathrm{SL}_d(\mathbb{R})^D$ and
$\Gamma_{\mathcal{O}} = \{(\sigma_1(\gamma), \ldots, \sigma_d(\gamma)) : \gamma \in \mathrm{SL}_d(\mathcal{O})\}$.

Proof.

- Define $\alpha(\mathcal{L}) = \max\left(\mathrm{covol}(\mathcal{L}')^{-1} : \mathcal{L}' < \mathcal{L}\right)$. Then $\hat{f} \ll \alpha$.
- If $\mathcal{L} = g\mathbb{Z}^n = kan\mathbb{Z}^n$, then $\alpha(\mathcal{L}) \ll \alpha(a\mathbb{Z}^n)$ where $a = \mathrm{diag}\,(a_1, \ldots, a_n)$ (via Siegel domain)
  or, if λ_i Minkowski's successive minima, then $\alpha(\mathcal{L}) \asymp \lambda_1 \cdot \cdots \cdot \lambda_{i_0}$, where i_0 last index such that $\lambda_{i_0} < 1$.
- Denote A_d diagonal matrices in $\mathrm{SL}_d(\mathbb{R})$

□

PROOF.

- Define $\alpha(\mathcal{L}) = \max\left(\text{covol}(\mathcal{L}')^{-1} : \mathcal{L}' < \mathcal{L}\right)$. Then $\hat{f} \ll \alpha$.
- If $\mathcal{L} = g\mathbb{Z}^n = kan\mathbb{Z}^n$, then $\alpha(\mathcal{L}) \ll \alpha(a\mathbb{Z}^n)$ where $a = \text{diag}\left(a_1, \ldots, a_n\right)$ (via Siegel domain)
  or, if λ_i Minkowski's successive minima, then $\alpha(\mathcal{L}) \asymp \lambda_1 \cdot \cdots \cdot \lambda_{i_0}$, where i_0 last index such that $\lambda_{i_0} < 1$.
- Haar measure on $\text{SL}_d(\mathbb{R})$ in KA_dN coordinates given by
  $dg = dk\ da\ \rho(a)\ dn$
  where $\rho(a) = |\det \text{Ad}(a)|_{\mathfrak{n}}| = \prod_{i<j} \frac{a_i}{a_j}$
  and $da = \prod_{i=1}^{d-1} \frac{da_i}{a_i}$
- Remains to find a fundamental domain (or rather surjective set) for $\Gamma_{\mathcal{O}}$.

□

Proof.

- Remains to find a fundamental domain (or rather surjective set) for $\Gamma_{\mathcal{O}}$: Follow Siegel domain proof for $\Gamma_{\mathbb{Z}}$.
- $\Gamma_{\mathcal{O}}$ stabilizer of the lattice $\mathcal{L}_{\mathcal{O}} = \{\sigma_1(v), \ldots, \sigma_D(v) : v \in \mathcal{O}^d\}$.
- Upper unipotents of $\Gamma_{\mathcal{O}}$ (uniform) lattice in upper unipotents of $\mathrm{SL}_d(\mathbb{R})^D$
- By Minkowksi successive minima + Gram Schmidt $g = kan$ with, $a_i \asymp \lambda_i(a\mathcal{L}_{\mathcal{O}})$, $a_{i+1}/a_i \gg 1$ in each each $\mathrm{SL}_d(\mathbb{R})$ block (so $i = jd \ldots ...(j+1)d - 1$).
- Have units in $\mathcal{O}$, hence have non-compact diagonal subgroup in $\Gamma_{\mathcal{O}}$.
  Can find $u \in \mathcal{O}^{\times}$ such that for any $z \in \mathbb{K}$, $|\sigma_i(zu)| \asymp N(z)^{1/D}$
- Can be used to find $\gamma \in \Gamma_{\mathcal{O}}$ such that for any $a \in A_d^D$,$a\gamma$ has $(a\gamma)_{jd} \asymp (a\gamma)_{1d}$ for $j = 1 \ldots D$

□

Proof.

- In summery: Instead of integrating over D-fold product of $(A_d)_c = \{\frac{a_{i+1}}{a_i} \geq c\}$, only need to integerate over **one** $(A_d)_c$ and $D-1$ many compact neighborhoods of identity in A_d.
- Have now explicit formula of integrand and measure, so an explicit calculation (identically to EMM99) remains, giving same integrability exponent as $\mathbb{Z}$ case.

□

- The probabilistic almost all counting result uses bounds on second moments (Borel-Cantelli)
- This is given by Rogers type formula:

Roger Type Formula

Suppose $h \in C_c(\mathbb{R}^{pn})$.

$$^p\widehat{h}(\mathcal{L}) = \sum_{v_1,\ldots,v_p} h(v_1, \ldots, v_p)$$

If $^p\widehat{h}(\mathcal{L}) \in L^1(\mu)$ for μ L-invariant homogeneous measure

$$\mu(^p\widehat{h}) = \sum \tau_i(h)$$

where τ_i are Lebesgue measures supported on L-orbits on $\mathbb{R}^{pn}$, and hence Lebesgue on some some hyperplane.

- Riesz representation Theorem (directly from integrability)
- Unfolding a la Weil (Something to check: finite volume)

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Rogers: $\mu = m_{X_n}$, $n > 2$. Macbeath-R Determines what *exactly* the τ_i are Generalizations by Yu, Kelmer-Yu, Ghosh-KY, Han to other groups, congruence lattice, S-adics... For application suffices to get bound on volume growth. Schmidt $n = 2$. See Kleinbock-Skenderi "Rogers type groups"

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If $^{p}\widehat{h}(\mathcal{L}) \in L^1(\mu)$ for μ L-invariant then

$$\mu(^{p}\widehat{h}) = \sum \tau_i(h)$$

where τ_i are Lebesgue measures supported on L-orbits on $\mathbb{R}^{pn}$, and hence Lebesgue on some some hyperplane.

Get useful moment bounds for Siegel transform of $f \in C_c(\mathbb{R}^n)$: $\mu(\widehat{f}^p) = \ m_{\mathbb{R}^n}(f)^p + \tau_{rest}(f^{\otimes p})$ with τ_{rest} lower order volume growth.

Let $G_1 \subset G_2$ unimodular, $\Gamma_2 \subset G_2$ lattice, $\Gamma_1 = \Gamma_2 \cap G_1$, and for any $\gamma \in \Gamma_2$, denote its coset $\gamma\Gamma_1 \in \Gamma_2/\Gamma_1$ by $[\gamma]$.

Weil Unfolding Theorem

Assume that Γ_1 is a lattice in G_1.
For any $F \in L^1(G_2/G_1, m_{G_2/G_1})$, define

$$\tilde{F}(g\Gamma_2) = \sum_{[\gamma] \in \Gamma_2/\Gamma_1} F(g\gamma).$$

Then $\tilde{F} \in L^1(G_2/\Gamma_2, m_{G_2/\Gamma_2})$ and

$$\int_{G_2/\Gamma_2} \tilde{F}\, dm_{G_2/\Gamma_2} = \int_{G_2/G_1} F\, dm_{G_2/G_1}.$$

$G = G_2$. $G_1 = \mathrm{Stab}(v, w)$, for $(v, w) \in \mathcal{O}^{2d}$.

In more detail: $m_{\mathrm{SL}_d(\mathbb{R})^D/\Gamma_{\mathcal{O}}}(\widehat{f}^2) = \sum \tau_i(f \otimes f) = m_{\mathbb{R}^{Dd}}(f)^2 + \mathcal{O}(m_{\mathbb{R}^{Dd}}(f))$

- Decompose pairs of vectors $\{(v, w) \in \mathcal{O}^d \times \mathcal{O}^d\}$ in $\mathrm{SL}_d(\mathcal{O})$-orbits.
- Sum over these single orbits correspond to "primitive Siegel transform"
- Unfolding identifies this discrete measure with measure on $\mathrm{SL}_d(\mathbb{R})^D$ modulo stabilizer of (v, w)
- Three types: $w = 0$, $v \in \mathbb{K}^\times w$, linear independent
- $\mathrm{SL}_d(\mathbb{K}) < \mathrm{SL}_d(\mathbb{K}_S) = \mathrm{SL}_d(\mathbb{R})^D$ transitive on linear independent vectors of $\mathcal{O}^d < \mathbb{R}^{dD}$. This is the main term, corresponding to $\tau_i = \tau_{\mathrm{main}} = m_{\mathbb{R}^{2dD}}$
  $\mathrm{stab}(v, w) = \mathrm{SL}_{d-2}(\mathbb{K}_S) \ltimes \mathbb{K}_S^{2(d-2)}$ up to a conjugation in $\mathrm{SL}_d(\mathbb{K})$. (namely, by some matrix having (v, w) as first two column vectors)
  Intersects $\mathrm{SL}_d(\mathcal{O})$ in a lattice. So Weil applies.

In more detail: $m_{\mathrm{SL}_d(\mathbb{R})^D/\Gamma_{\mathcal{O}}}(\widehat{f}^2) = \sum \tau_i(f \otimes f) = m_{\mathbb{R}^{Dd}}(f)^2 + \mathcal{O}(m_{\mathbb{R}^{Dd}}(f))$

- Other orbits (av, bv), for $a, b \in \mathbb{K}$.
- Plugging in (product) balls, can calculate explicitely when integrating against "dv" since correlation of two scaled balls is again a ball.
- Can move to general functions by using Rogers / Brasscamp-Lieb-Luttinger symmetrization formula (see discussions in Skenderi)
- Alternatively, do a trick of Schmidt $f(x)f(\frac{a}{b}x) \leq f(\frac{a}{b}x)$ for f characteristic function.
- In any case, need some care of showing $\sum c_i < \infty$ (follows from integrability, i.e. the abstract Rogers formula).
- Caveat: f must be a product function $f_1 \ldots f_D$ resp. get Roger type bound only for the RMS measure, not general homogenous orbit.

Theorem (Rogers, Brascamp-Lieb-Luttinger])

Suppose $f_j : \mathbb{R}^d \to \mathbb{R}$, $j = 1, \ldots, \ell$ *are nonnegative, and* $\{a_{jm}\}$ *an* $\ell \times r$*-matrix. Then*

$$\int_{(\mathbb{R}^d)^r} f_1(\sum a_{1m}x_m) \ldots f_\ell(\sum a_{\ell m}x_m) dm_{\mathbb{R}^d}(x_1) \ldots dm_{\mathbb{R}^d}(x_r)$$

$$\leq \int_{(\mathbb{R}^d)^r} f_1^*(\sum a_{1m}x_m) \ldots f_\ell^*(\sum a_{\ell m}x_m) dm_{\mathbb{R}^d}(x_1) \ldots dm_{\mathbb{R}^d}(x_r)$$

where given a function $f : \mathbb{R}^d \to \mathbb{R}$, *the function* $f^* : \mathbb{R}^d \to$ *is the symmetric decreasing rearrangement, i.e. if* $f = \mathbb{1}_A$ *then* $f^* = \mathbb{1}_{B_r(0)}$ *where* $|A| = |B_r(0)|$

- Up to conjugation in $\mathrm{SL}_d(\mathbb{K})$, stabilizer of $\mathrm{SL}_d(\mathbb{K}_S)$ at (av, bv), $v \in \mathcal{O}^d$, $a, b \in \mathbb{K}$ is $\mathrm{SL}_{d-1}(\mathbb{K}_S) \ltimes \mathbb{K}_S^{d-1}$
- $\mathrm{SL}_d(\mathbb{K}_S)$-orbit at $\mathfrak{e} = (av, bv)$ in $\mathbb{K}_S^{2d}$ is a ($\mathbb{K}$-rationally) skewed $\mathbb{K}_S^d$ with Lebesgue measure being a multiple $c_{\mathfrak{e}}$ of a push-forward of a fixed Haar measure $(\times a, \times b)_* m_{\mathbb{K}_S^d}$
- $\tau_{\text{rest}}(f \otimes f) = \sum_{\mathfrak{e}} c_{\mathfrak{e}} (\times a, \times b)_* m_{\mathbb{K}_S^d}(f \otimes f)$
- $(\times a, \times b)_* m_{\mathbb{K}_S^d}(f \otimes f) = \int f(aw) f(bw) dm_{\mathbb{K}_S^d}(w)$

$$= \int_{\mathbb{R}^{dD}} f_1(\sigma_1(a)x_1) \dots f_D(\sigma_D(a)x_D) f_1(\sigma_1(b)x_1) \dots f_D(\sigma_d(b)x_D)$$

$$dm_{\mathbb{R}^d}(x_1) \dots dm_{\mathbb{R}^d}(x_D)$$

- Apply symmetrization to $\ell = 2D$, $r = D$

- By the previous, if $f = \prod \mathbb{1}_{A_i}$, $|A_i| = V_i = |B_{r_i}(0)|$

$$\tau_{\text{rest}}(f \otimes f)$$

$$\leq \sum_{\mathfrak{e}} c_{\mathfrak{e}} \int_{\mathbb{R}^{dD}} \prod \mathbb{1}_{B_{r_i}(0)}(\sigma_i(a)x_i)\mathbb{1}_{B_{r_i}(0)}(\sigma_i(b)x_i) dm_{\mathbb{R}^d}(x_i)$$

$$= \sum c_{\mathfrak{e}} \prod \max(|\sigma_i(a)|, |\sigma_i(b)|)^{-d} V_i = c_{\text{rest}} m_{\mathbb{K}_S^d}(f)$$

- $c_{\text{rest}} < \infty$ since the inequality $\leq$ is equality if f is a product of balls, and $\tau_{\text{rest}}(f \otimes f) < \infty$

- Got Rogers bound for "product sets " $A_1 \times \dots A_D$
- This suffices for the application, which needs $A \times \mathcal{W}$ where $|A| \to \infty$.
- Indeed, take $A_2 \times \cdots \times A_D \supset \mathcal{W}$ to dominate in τ_{rest}. These are fixed, hence absorbed into the constant.