Low Entropy Method (following "Invariant measures and arithmetic quantum unique ergodicity", Lindenstrauss 2006)

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Outline

Statement of Theorem

G^{-}-invariance implies $\mathrm{SL}_2(\mathbb{R})$-invariance

Outline

Leafwise is Haar zero-one reduction

T-leaves and T-recurrence

Break and Recap)

H-property

Maximal Ergodic Theorem

G_a^{-}-Equivariance and T-Invariance

Continuity

Doubling Condition

Collecting good properties in X_1

Proof using Doubling Condition

Doubling radius for H-pairs imply Equal leafwise measures

Removing the Additional Assumption

Entropy implies Weak Doubling

Maximal Ergodic Theorem - Second Entrance

Construction of H-pairs

Products with $\mathrm{SL}_2(\mathbb{R})$

$G = \mathrm{SL}_2(\mathbb{R}) \times T$

T= S-algebraic group (i.e. $T = \mathbb{G}(\mathbb{Q}_S)$)

Γ= discrete subgroup of G s.t. $T \cap \Gamma$ finite.

X= $(\mathrm{SL}_2(\mathbb{R}) \times T)/\Gamma$

A= diagonals in $\mathrm{SL}_2(\mathbb{R})$

Example

$$X = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)/\,\mathrm{SL}_2(\mathbb{Z}[\tfrac{1}{p}])$$

Recurrence

$G = \mathrm{SL}_2(\mathbb{R}) \times T$

T= S-algebraic group

Γ= discrete subgroup of G s.t. $T \cap \Gamma$ finite.

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A= diagonals in $\mathrm{SL}_2(\mathbb{R})$

μ Probability

Invariant A-inv. probability on X

Recurrent T-recurrent

Entropy $\mu = \int \mu_x^{\mathcal{E}} d\mu$ ergodic decomp. then $h_{\mu_x^{\mathcal{E}}}(a) > 0$ a.e.

Recurrence

For any $B \in \mathcal{B}(X)$, μ-a.e. $x \in B$, $\{t \in T : t.x \in B\}$ unbounded.

Recurrence vs Entropy

$G = \mathrm{SL}_2(\mathbb{R}) \times T$

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μ Probability

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Entropy $\mu = \int \mu_x^{\mathcal{E}} d\mu$ ergodic decomp. then $h_{\mu_x^{\mathcal{E}}}(a) > 0$ a.e.

Entropy Assumption true iff

Entropy G_a^--recurrent

Main Theorem

$G = \mathrm{SL}_2(\mathbb{R}) \times T$

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Γ= discrete subgroup of G s.t. $T \cap \Gamma$ finite.

X= $(\mathrm{SL}_2(\mathbb{R}) \times T)/\Gamma$

A= diagonals in $\mathrm{SL}_2(\mathbb{R})$

μ PROBABILITY

INVARIANT A-inv. probability on X

RECURRENT T-recurrent

ENTROPY $\mu = \int \mu_x^{\mathcal{E}} d\mu$ ergodic decomp. then $h_{\mu_x^{\mathcal{E}}}(a) > 0$ a.e.

THEOREM (LINDENSTRAUSS 2006)

μ is convex combination of $\mathrm{SL}_2(\mathbb{R})$-inv. algebraic measures $m_{H_x.x}$

Special Case Theorem

$X = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p) / \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$

μ Probability

Invariant A-inv. probability on X

Recurrent $\mathrm{SL}_2(\mathbb{Q}_p)$-recurrent

Entropy $\mu = \int \mu_x^{\mathcal{E}} d\mu$ ergodic decomp. then $h_{\mu_x^{\mathcal{E}}}(a) > 0$ a.e.

Theorem (Lindenstrauss 2006)

μ is Haar

μ is $\mathrm{SL}_2(\mathbb{R})$-invariant and since any $\mathrm{SL}_2(\mathbb{R})$-orbit is dense ($\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]) < \mathrm{SL}_2(\mathbb{Q}_p)$ is dense) it must be Haar

Unstable/Stable groups

Welcome Applause for our Actors

$$a = a_t = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix} \qquad G_a^- = \{u_s = \left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right]\}$$

Theorem (Lindenstrauss 2006)

Invariant μ *A-inv. probability on* $X = \mathrm{SL}_2(\mathbb{R}) \times T/\Gamma$

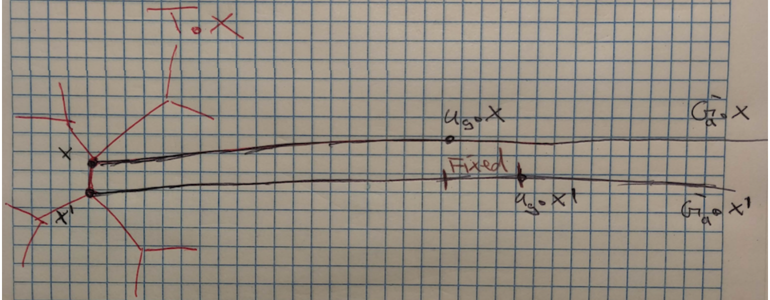
Recurrent μ *is T-recurrent*

Entropy $\mu = \int \mu_x^{\mathcal{E}} d\mu$ *ergodic decomp. then* $h_{\mu_x^{\mathcal{E}}}(a) > 0$ *a.e.*

Then μ *is* G_a^-*-invariant.*

Proof of Main Theorem.

By an involution argument, μ is also $G_a^+ = \{n_s = \left[\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right]\}$-invariant. Hence $< G_a^-, G_a^+ >= \mathrm{SL}_2(\mathbb{R})$-invariant. Ratner/Margulis-Tomanov implies that μ is algebraic. □



Take $d(x_n, x_n') \to 0$

$x_n' \in T\cdot x_n$

$$\lim u_{g_n}\cdot x_n' = \ddot{u}\cdot \lim u_{g_n}\cdot x_n$$

$$1 \ll \|\ddot{u}\| \ll 1$$

$$\boxed{\mu_x^{\bar{G}_a} \overset{x'\in T\cdot x}{\equiv} \mu_{x'}^{\bar{G}_a}}$$

$$\lim \mu^{\bar{G}_a}_{u_{g_n}\cdot x_n'} = \ddot{u}\cdot \lim \mu^{\bar{G}_a}_{u_{g_n}\cdot x_n}$$

$$\parallel$$

$$\lim \mu^{\bar{G}_a}_{u_{g_n}\cdot x_n}$$

$$\Rightarrow \mu^{\bar{G}_a}_{\lim u_{g_n}\cdot x_n} \quad \ddot{u}\text{-invariant}$$

Leafwise is Haar zero-one reduction

Definition

$$X_{\mathrm{Haar}} = \{x : \mu_x^{G_a^-} = \mathrm{Haar}\}$$

Lemma (Reduction Lemma)

If the theorem fails (i.e. $\mu(X_{Haar}) < 1$) then we may assume $\mu(X_{Haar}) = 0$.

Hopf property for Leafwise Measures

Suppose $Z = X_{\mathrm{Haar}}$ has $\mu(Z) < 1$.

Lemma

For any A-invariant Z s.t. $\mu(Z) < 1$, then $\mu' = \mu|_{Z^c}$ is A-invariant and for μ'-a.e. x

$$\mu_x'^{G_a^-} = \mu_x^{G_a^-}$$

Proof.

Hopf argument: Z^c is G_a^--invariant. (Pisa 7.20)
Leafwise measure construction: $(\mu_x^{\mathcal{A}})_y^{G_a^-} = \mu_y^{G_a^-}$ if $\mathcal{A}$ a-inv.
Take $\mathcal{A} = \mathcal{B}_{Z^c}$. □

Proof of Haar zero-one reduction

Suppose $Z = \{x : \mu_x^{G_a^-} = \text{Haar}\}$ has $\mu(Z) < 1$.

Lemma

For any A-invariant Z s.t. $\mu(Z) < 1$, then $\mu' = \mu|_{Z^c}$ is A-invariant and for μ'-a.e. x

$$\mu_x'^{G_a^-} = \mu_x^{G_a^-}$$

Proof of Haar zero-one reduction.

μ' still T-recurrent (since Z^c pos measure).
All ergodic components are positive entropy (since Z^c A-invariant).
So if theorem fails for μ it fails for μ' (same leafwise measures).
Hence "Z for μ'" has $\mu'(Z) = 0$. □

Weaken Z to equivalence classes

Lemma

$$X_{\textit{Some-Inv}} = \{x : [\mu_x^{G_a^-}] = [\mu_x^{G_a^-} u_s] \textit{ for some } u_s\}$$

is conull in $X_{\textit{Haar}} = \{x : \mu_x^{G_a^-} = \textit{Haar}\}$.

Hence if theorem fails, we may assume $\mu(X_{\text{Some-Inv}}) = 0$.

Weaken Z to equivalence classes

Lemma

$$Y = X_{Some\text{-}Inv} = \{x : [\mu_x^{G_a^-}] = [\mu_x^{G_a^-} u_s] \text{ for some } u_s\}$$

is conull in X_{Haar}.

Proof.

- By Poincare recurrence (on $\{x : [\mu_x^{G_a^-}] = [\mu_x^{G_a^-} u_s], |s| \leq C\}$) $\{x : [\mu_x^{G_a^-}] = [\mu_x^{G_a^-} u_s]$ for arbitraily small $u_s\}$ conull in Y.
- For $\phi \in C_c(G_a^-)$ define $s \mapsto \kappa(s, x) = \log \frac{u_{s*}\mu_x^{G_a^-}(\phi)}{\mu_x^{G_a^-}(\phi)}$.
- Since $\mu_x^{G_a^-} u_s = c\mu_x^{G_a^-}$ implies $\mu_x^{G_a^-} u_{ns} = c^n \mu_x^{G_a^-}$ we have for any s, any n, m, $\exists s' \overset{\varepsilon}{\sim} s$, $\kappa(\frac{n}{m}s', y) = \frac{n}{m}\kappa(s', y)$.
- By continuity in s, $\kappa(s, y) = s\kappa(1, y)$ for any s.
- $[a_{t*}\mu_y^{G_a^-}] = [\mu_{a_t y}^{G_a^-}]$ so $\kappa(s, a_t y) = \kappa(e^{-2t}s, y) = e^{-2t}\kappa(s, y) \equiv 0$ (Poincare)

T-leaves are embedded

Finite index reduction

Assumption: $\Gamma \cap T$ finite.
Find $\Gamma' < \Gamma$ normal finite index subgroup with $\Gamma' \cap T = \{e\}$
Theorem for $(\mathrm{SL}_2(\mathbb{R}) \times T)\,/\Gamma'$ implies statement for X by lifting μ.
So may assume $\Gamma \cap T = \{e\}$.

$T.x$ is embedded

Then $t \mapsto t.x$ injective since if $t.x = x = g_1 g_T \Gamma$ then
$\Gamma = g_T^{-1} t g_T \Gamma$ so $g_T^{-1} t g_T \in \Gamma \cap T$.

$G_a^- \times T$-leaves embedded almost surely

Suppose $u_s \in G_a^-$: $u_s t.x = x$

Poincare recurrence on compact subset:

There is $r_n \to \infty$ and x_0 s.t. $a_{r_n}.x \to x_0$. Since $G_a^- = G_{a_{r_n}}^-$,

$$x_0 = \lim a_{r_n}.x = \lim a_{r_n} u_s t.x = t.x_0.$$

Contradiction to T being embedded.

Embedded T-leaves and T-recurrence vs G_a^--orbits

Lemma (T-Leaf Lemma)

For any $\varepsilon > 0$, any $B \in \mathcal{B}_X$, μ-a.e. $x \in B$

$$T.x \cap B \cap B_\varepsilon(x) \setminus B_1^{G_a^- \times T}(x) \quad \textit{is non-empty}$$

Proof.

Let $B_{\varepsilon/2}(x_i)$ countable cover of B.
By T-recurrence, exists $t_n \to \infty$ s.t.
$t_n.x \in B \cap B_{\varepsilon/2}(x_i) \subset B \cap B_\varepsilon(x)$ for a.e. $x \in B \cap B_{\varepsilon/2}(x_i)$. $\square$

Pesach Break

Theorem and Reduction

Notation

$$A = \left\{ a = a_t = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \right\} \qquad G_a^- = \{ u_s = \left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right] \}$$

Theorem (Lindenstrauss 2006)

Invariant μ *A-inv. probability on* $X = \mathrm{SL}_2(\mathbb{R}) \times T/\Gamma$

Recurrent μ *is T-recurrent*

Entropy $\mu = \int \mu_x^{\mathcal{E}} d\mu$ *ergodic decomp. then* $h_{\mu_x^{\mathcal{E}}}(a) > 0$ *a.e.*

Then μ *is* G_a^-*-invariant.*

Lemma (Reduction)

$$X_{Some\text{-}Inv} = \{ x : [\mu_x^{G_a^-}] = [\mu_x^{G_a^-} u_s] \text{ for some } u_s \}$$

If μ *is not* G_a^-*-invariant then we may assume* $\mu(X_{Some\text{-}Inv}) = 0$.

T-recurrence

Lemma (T-Leaf Lemma)

For any $\varepsilon > 0$, *any* $B \in \mathcal{B}_X$, μ-*a.e.* $x \in B$

$$\exists x' \in T.x \cap B \cap B_\varepsilon(x) \text{ but } x' \notin B_1^{G_a^- \times T}(x)$$

H-property

Lemma (Ratner '83 (Lindenstrauss '06))

$$x' = g.x, \; g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \; d(g,e) = \max(|a-1|, |b|, |c|) \leq \varepsilon$$

For any ρ, there is C so that for any $x, x' = g.x$ there is some $r = r_g$ so that for either for any $s \in [\rho r, r]$ or any $s \in [-\rho r, -r]$ there is $1/C \leq |\sigma| \leq C$ s.t.

$$u_s.x' = (u_s g u_{-s}).u_s.x = g'.u_\sigma u_s.x \quad \text{for some } d(g', e) \leq 2\varepsilon^{1/2}.$$

Proof of H-property. $\|g\| < \varepsilon$

Calculate:

$$u_s.x' = g_s.u_s.x, \qquad g_s = u_s g u_{-s} = \begin{bmatrix} a + cs & b + (d-a)s - cs^2 \\ c & d - cs \end{bmatrix}$$

Define:

$$r = \min\left(\frac{C^{1/2}}{|c|^{1/2}}, \frac{C}{|d-a|}\right), \qquad C = \rho^{-1}$$

Claim: For either $s \in [\rho r, r]$ or $s \in [-\rho r, -r]$

$$g_s \in B^{\mathrm{SL}_2(\mathbb{R})}_{\varepsilon^{1/2}} u_{p(s)}, \qquad p(s) = (d-a)s - cs^2 \in [1/C, 2C]$$

Case 1: $r = (C/|c|)^{1/2}$.
If $\operatorname{sign}(s) = \operatorname{sign}((a-d)c)$ then for $|s| \in [\rho r, r]$

$$|p(s)| \leq C^{1/2}\frac{|d-a|}{|c|^{1/2}} + C \leq 2C$$

$$|p(s)| \geq \rho C^{1/2}\frac{|d-a|}{|c|^{1/2}} + \rho^2 C \geq \rho^2 C = \frac{1}{C}$$

Proof of H-property. $\|g\| < \varepsilon$

Calculate:

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$$g_s \in B^{\mathrm{SL}_2(\mathbb{R})}_{\varepsilon^{1/2}} u_{p(s)}, \qquad p(s) = (d-a)s - cs^2 \in [1/C, 2C]$$

Case 2: $r = \frac{C}{|d-a|}$.

If $\mathrm{sign}(s) = \mathrm{sign}\left((a-d)c\right)$ then for $|s| \in [\rho r, r]$, and $C = \rho^{-1}$

$$|p(s)| \leq C + C\frac{|c|}{|d-a|^2} \leq 2C$$

$$|p(s)| \geq \rho C + \rho^2 C\frac{|c|}{|d-a|^2} \geq \rho C \geq \frac{1}{C}$$

Notation: Points with the H-Property

Definition (Good annuli for H-pairs)

$$\mathsf{H}_{\rho,C,\varepsilon'}(x,x') = \{r : \forall s \in \pm[\rho r, r]\ \exists|\sigma| \in [1/C, C] : u_s.x' \in B_{\varepsilon'}(u_\sigma u_s.x)\}$$

We just proved:

Lemma (H-Property)

If $x' \in B_\varepsilon(x) \setminus B_1^{G_a^-}.x$ *then* $H_{\rho,\frac{1}{\rho},\varepsilon^{1/2}}(x,x') \neq \emptyset$

Maximal Ergodic Theorem for non-invariant Actions

Theorem (Lindenstrauss-Rudolph)

Define $M_\mu(f)(x) = \sup_{r>0} \frac{1}{\mu_x^{G_a^-}(B_r^{G_a^-})} \int_{B_r^{G_a^-}} |f(u_s.x)| d\mu_x^{G_a^-}(u_s)$.

Then

$$\mu(x : M_\mu(f)(x) > R) \leq c\frac{\|f\|_1}{R}.$$

Corollary (Suppose $\mu(X') > 1 - \varepsilon$.)

There is $X_{u\text{-}Erg(X')} \subset X'$, $\mu(X_{u\text{-}Erg(X')}) > 1 - c\varepsilon^{1/2}$ *s.t.*

$$\int_{B_r^{G_a^-}} 1_{X'}(u_s.x) d\mu_x^{G_a^-}(u_s) \geq (1 - \varepsilon^{1/2})\mu_x^{G_a^-}(B_r^{G_a^-})$$

for all $x \in X_{u\text{-}Erg(X')}$, *for all* $r > 0$.

Proof of Corollary.

$f = 1 - 1_{X'}$. $R = \varepsilon^{1/2}$. $X_{u\text{-}\mathrm{Erg}(X')} = X \setminus \{x : M_\mu(f) > \varepsilon^{1/2}\}$ □

Maximal Ergodic Theorem: Proof

Lemma

For all $\varepsilon, r > 0$ there is a G_a^--subordinate a-descending σ-algebra $\mathcal{A}$ s.t.

$$[x]_{\mathcal{A}} \subset G_a^-.x$$

and

$$\mu(\{x : B_r^{G_a^-}.x \subset [x]_{\mathcal{A}}\}) > 1 - \varepsilon.$$

Proof.

Let $\mathcal{A}$ be a G_a^--subordinate a-descending σ-algebra, in particular $B_\delta^{G_a^-}(x) \subset [x]_{\mathcal{A}}$ for some $\delta = \delta_x$. Take δ_0 s.t. $Z = \{x : \delta > \delta_0\}$ has mass $> 1 - \varepsilon$. Then $a^{-n}\mathcal{A}$ on $a^{-n}Z$ has atoms of size $e^{2n}\delta_0 > r$. □

Maximal Ergodic Theorem: Proof

Definition

$$M_{\mu,r}(f)(x) = \sup_{r>\rho>0} \frac{1}{\mu_x^{G_a^-}(B_\rho^{G_a^-})} \int_{B_\rho^{G_a^-}} |f(u_s.x)| d\mu_x^{G_a^-}(u_s).$$

Proof.

$Y = \{x : M_\mu(f)(x) > R\}$, $Y' = \{x : M_{\mu,r}(f)(x) > R/2\}$.
Then $\mu(Y') > \mu(Y)/2$ for r large.
Lemma: $\exists X' = \{B_r^{G_a^-}.x \subset [x]_{\mathcal{A}}\}$ such that $\mu(X') > 1 - \mu(Y)/4$.
Then $Y'' = X' \cap Y'$ has $\mu(Y'') \geq \mu(Y)/4$.
Note also:

$$M_{\mu,r}(f)(x) = \sup_{r>\rho>0} \frac{1}{\mu_x^{\mathcal{A}}(B_\rho^{G_a^-}.x)} \int_{B_\rho^{G_a^-}.x} |f(z)| d\mu_x^{\mathcal{A}}(z).$$

□

Maximal Ergodic Theorem: Proof

$$Y'' = X' \cap Y' = \left\{ B_r^{G_a^-}.x \subset [x]_{\mathcal{A}} \right\} \cap \left\{ x : M_{\mu,r}(f)(x) > R/2 \right\}$$

Proof.

$\forall y \in Y_x := Y'' \cap [x]_{\mathcal{A}}$ there is $r_y < r$ s.t.

$$\int_{B_{r_y}^{G_a^-}.y} |f(z)| d\mu_x^{\mathcal{A}}(z) > \frac{R}{2} \mu_x^{\mathcal{A}}(B_{r_y}^{G_a^-}.y).$$

Since $B_{r_y}^{G_a^-}.y \subset [x]_{\mathcal{A}}$, $\{B_{r_y}^{G_a^-}.y\}$ defines a cover of Y_x. By Besicovitch Covering Thm $\exists$ countable subcover $\{B_{r_{y_n}}^{G_a^-}.y_n\}$ with $\leq C = C(G_a^-)$ overlaps over any point.

$$\int_{[x]_{\mathcal{A}}} |f(z)| d\mu_x^{\mathcal{A}}(z) \geq \frac{1}{C} \sum_n \int_{B_{r_{y_n}}^{G_a^-}.y_n} |f(z)| d\mu_x^{\mathcal{A}}(z)$$

□

Maximal Ergodic Theorem: Proof

$Y_x := Y'' \cap [x]_{\mathcal{A}}$, $Y'' = X' \cap Y'$, $\mu(Y'') \geq \mu(Y)/4$,

Proof.

$$\int_{[x]_{\mathcal{A}}} |f(z)| d\mu_x^{\mathcal{A}}(z) \geq \frac{1}{C} \sum_n \int_{B_{r_{y_n}}^{G_a^-} . y_n} |f(z)| d\mu_x^{\mathcal{A}}(z)$$

$$> \frac{R}{2C} \sum_n \mu_x^{\mathcal{A}}(B_{r_{y_n}}^{G_a^-} . y_n) \geq \frac{R}{2C} \mu_x^{\mathcal{A}}(Y_x)$$

Integrate w.r.t. μ:

$$\|f\|_1 = \mu(|f|) > \frac{R}{2C} \mu(Y'') \geq \mu(Y) R/8C$$

□

Leafwise Measure properties

Equivariance of origin

There is $X_{G_A^- \text{-Equi}} \subset X$ co-null such that for $x, u_s.x \in X_{G_A^- \text{-Equi}}$

$$[\mu_x^{G_a^-}] = [\mu_{u_s.x}^{G_a^-}.u_s]$$

Normalization

May assume $\mu_x^{G_a^-}(B_1^{G_a^-}) = 1$ for all $x \in X_{G_A^- \text{-Equi}}$.

Product structure for $G_a^- \times T$ implies

There is $X_{T\text{-Inv}} \subset X$ co-null such that for $x, y \in X_{T\text{-Inv}} \cap T.x$

$$[\mu_x^{G_a^-}] = [\mu_y^{G_a^-}].$$

Continuity of leafwise measures

Lemma (Lusin)

For all $\varepsilon > 0$ there exists $X_{Cont} \subset \{x \mapsto [\mu_x^{G_a^-}]$ continuous $\}$ s.t. X_{Cont} is compact and $\mu(X_{Cont}) > 1 - \varepsilon$.

Uniform Integrability (Pisa 6.30)

Let $b_n > 0$ be summable and $r_n \to \infty$. Then $\rho(u) = \frac{b_n^2}{m_{G_a^-}(B_{r_n+5}^{G_a^-}(u))}$

for $u \in B_{r_n}^{G_a^-} \setminus B_{r_{n-1}}^{G_a^-}$ is piecewise constant, strictly positive and $\rho \in L^1(\mu_x^{G_a^-})$ for μ-a.e. x.

Metric

$\{f_n\} \subset C_c(G_a^-) \cap \{f \leq \rho\}$ dense,

$$d([\mu_x^{G_a^-}], [\mu_y^{G_a^-}]) = \sum_n 2^{-n} \left| \frac{\mu_x^{G_a^-}(f_n)}{\mu_x^{G_a^-}(\rho)} - \frac{\mu_y^{G_a^-}(f_n)}{\mu_y^{G_a^-}(\rho)} \right|$$

Doubling Condition

The H-Property gives us an annulus $\{u_s : |s| \in [\rho r, r]\}$. How to make sure that modulo $\mu_x^{G_a^-}$, $\{u_s : |s| \in [0, \rho r]\}$ is negligible?

Definition (Doubling radii)

$$\mathcal{R}_\rho(x) := \left\{ r : \mu_x^{G_a^-}(B_r^{G_a^-}) > 2\mu_x^{G_a^-}(B_{\rho r}^{G_a^-}) \right\}$$

Additional Assumption

There exists $\rho \in (0, 1)$

$$X_{\rho,\mathbb{R}\text{-Doubling}} = \{x : \mathcal{R}_\rho(x) = \mathbb{R}_{>0}\}$$

has $\mu(X_{\rho,\mathbb{R}\text{-Doubling}}) = 1$

Doubling Condition: Entropy implies Recurrence

What do we actually now about $\mathcal{R}_\rho(x)$ (instead of the Additional Assumption)?

Theorem (Pisa 6.26 & 7.7)

μ is G_a^--recurrent iff $\mu_x^{G_a^-}$ is infinite a.e. iff $\mu_x^{G_a^-}$ not trivial a.e. iff $D_\mu(a, G_a^-)(x) > 0$ a.e. iff $h_{\mu_x^{\mathcal{E}}}(a) > 0$ a.e.

Assumption

Entropy μ is G_a^--recurrent

Doubling Condition: Entropy implies Weak Doubling

Theorem

$\mu_x^{G_a^-}$ not trivial a.e. iff $h_{\mu_x^{\mathcal{E}}}(a) > 0$ a.e.

Corollary (and Definition)

$$X_{\rho,\text{1-Doubling}} := \left\{x : \mu_x^{G_a^-}(B_1^{G_a^-}) > 2\mu_x^{G_a^-}(B_\rho^{G_a^-})\right\} = \{x : 1 \in \mathcal{R}_\rho(x)\}$$

For all $\varepsilon > 0$ there is $\rho > 0$ such that

Entropy $\mu(X_{\rho,\text{1-Doubling}}) > 1 - \varepsilon$.

Proof.

$\mu_x^{G_a^-}(B_1^{G_a^-}) \equiv 1$ a.e. and $\mu_x^{G_a^-}(B_\rho^{G_a^-}) \to_{\rho\to 0} 0$ a.e. □

Again, for now assume that $\mu(X_{\rho,\mathbb{R}\text{-Doubling}}) = 1$

Nice set of large measure

$\forall \varepsilon > 0$ LET $X_1 \subset\subset X$, $\mu(X_1) > 1 - \varepsilon$ SUCH THAT

X-1 $X_1 \subset X^c_{\text{Some-Inv}}$

X-2 $X_1 \subset X_{\text{Cont}}$

X-3 $X_1 \subset X_{T\text{-Inv}}$

X-4 $X_1 \subset X_{G_A^-\text{-Equi}}$

X-5 $X_{u\text{-Erg}(X_1)} \subset X_1$ s.t. $\mu(X_{u\text{-Erg}(X_1)}) > 1 - \varepsilon^{1/2}$

X-9000 $X_1 \subset X_{\rho,\mathbb{R}\text{-Doubling}}$

Theorem (Lindenstrauss 2006)

Invariant μ *A-inv. probability on* $X = \mathrm{SL}_2(\mathbb{R}) \times T/\Gamma$

Recurrent μ *is T-recurrent*

Entropy $\mu = \int \mu_x^{\mathcal{E}} d\mu$ *ergodic decomp. then* $h_{\mu_x^{\mathcal{E}}}(a) > 0$ *a.e.*

Then μ *is* G_a^-*-invariant.*

Proof.

Goal: Construct $z \in X_1$ and show $z \in X_{\text{Some-Inv}}$.

Contradiction! $\square$

Combining H-property and Doubling Radius

Lemma (Equal Leafwise Measures Lemma)

For any $C, \delta > 0$, $x, x' \in X_{u\text{-}Erg(X_1)} \cap T.x$ *if*

$$H_{\rho,C,\varepsilon'}(x,x') \cap \mathcal{R}_\rho(x) \neq \emptyset$$

then there is $s \in \mathbb{R}$, $|\sigma| \in [1/C, C]$ *so that*

1) $y = u_s.x$, $y' = u_s.x' \in X_1$
2) $y' \in B_{\varepsilon'}(u_\sigma.y)$
3) $\mu_y^{G_a^+} = \mu_{y'}^{G_a^+}$

Proof.

3) 1) $\Rightarrow$ 3): $y, y' \in X_1 \subset X_{T\text{-Inv}} \cap X_{T\text{-Equi}}$ and since $y' = u.x' \in uT.x = T.y$ we have $\mu_y^{G_a^+} = \mu_{y'}^{G_a^+}$.
2) In 1) take $|s| \in [\rho r, r]$ for $r \in \mathrm{H}_{\rho,C,\varepsilon'}(x,x') \cap \mathcal{R}_\rho(x)$.

□

Combining H-property and Doubling Radius

Claim: $x, x' \in X_{u\text{-Erg}(X_1)}$, for any $r \in \mathcal{R}_\rho(x)$:

$$\mu_x^{G_a^-}(B_r^{G_a^-} \setminus B_{\rho r}^{G_a^-} \cap G_x \cap G_{x'}) > 0$$

where G_x, $G_{x'}$ return times to X_1 under u-flow starting at x, x'.

Proof of Lemma given Claim.

1) Let $s \in B_r^{G_a^-} \setminus B_{\rho r}^{G_a^-} \cap G_x \cap G_{x'}$ then $y = u_s.x$, $y' = u_s.x' \in X_1$.

□

Proof of Claim: Doubling Radius and Maximal Inequality

$x, x' \in X_{u\text{-}\mathrm{Erg}(X_1)}$ If $G_z = \{u : u.z \in X_1\}$ for $z = x, x'$ then

$$\mu_x^{G_a^-}(B_r^{G_a^-} \setminus B_{\rho r}^{G_a^-} \setminus G_z) \leq \mu_x^{G_a^-}(B_r^{G_a^-} \setminus G_z) \leq \varepsilon^{\frac{1}{2}} \mu_x^{G_a^-}(B_r^{G_a^-})$$

(by Maximal Ergodic Theorem and $\mu_x^{G_a^-} = \mu_{x'}^{G_a^-}$).

$r \in \mathcal{R}_\rho(x)$ $\mu_x^{G_a^-}(B_r^{G_a^-}) > 2\mu_x^{G_a^-}(B_{\rho r}^{G_a^-})$ and therefore

$$\mu_x^{G_a^-}(B_r^{G_a^-} \setminus B_{\rho r}^{G_a^-}) > \frac{1}{2} \mu_x^{G_a^-}(B_r^{G_a^-})$$

Combining these

$$\mu_x^{G_a^-}\left(\left(B_r^{G_a^-} \setminus B_{\rho r}^{G_a^-}\right) \cap G_x \cap G_{x'}\right) > 0$$

(since complement has relative measure $\leq 4\varepsilon^{1/2}$).

Proof of Theorem assuming Additional Assumption

Proof.

Let $\delta > 0$. Let $x \in X_1$.

By T-Leaf Lemma $\exists x' \in T.x \cap B_\delta(x) \cap X_{u\text{-Erg}(X_1)} \setminus B_1^{G_a^- \times T}(x)$

By H-Property Lemma $H(\rho, \rho^{-1}, \delta^{1/2})(x, x')$ non-empty.

Additional Assumption $\mathcal{R}_\rho(x) = \mathbb{R}_{>1}$.

By Equal Leafwise Measures Lemma:

There is s s.t. $y = u_s.x$, $y' = u_s.x'$, $y, y' \in X_1$, $\mu_y^{G_a^-} = \mu_{y'}^{G_a^-}$

$$y' \in B_{C\delta^{1/2}}(u_\sigma.y) \quad \text{for some} \quad u_\sigma \in B_C^{G_a^-} \setminus B_{1/C}^{G_a^-}$$

Apply Continuity of Leaswise measures.

PROOF.

Let $\delta_i \to 0$. Get $y_i, y_i' \in X_1 \subset X_{\text{Cont}}$
By compactness, may assume

$$y_i \to z \qquad y_i' \to z' \in \bigcap B_{C\delta_i^{1/2}}(u_{\sigma_i} y_i) = \text{ü}.z$$

for some $\text{ü} = \lim u_{\sigma_i} \in B_C^{G_a^-} \setminus B_{1/C}^{G_a^-}$ and $z, z' \in X_1$.
Since

$$\mu_{y_i}^{G_a^-} = \mu_{y_i'}^{G_a^-}$$

we find

$$\mu_z^{G_a^-} = \mu_{\text{ü}.z}^{G_a^-} = c_{\text{ü}.z} \mu_z^{G_a^-}.\text{ü}^{-1}$$

and hence the contradiction

$$z \in X_1 \cap X_{\text{Some-Inv}}.$$

□

Remove Additional Assumption

Too much

$$X_{\rho,\mathbb{R}\text{-Doubling}} =$$

$$\left\{x : \mathcal{R}_\rho(x) := \left\{r : \mu_x^{G_a^-}(B_r^{G_a^-}) > 2\mu_x^{G_a^-}(B_{\rho r}^{G_a^-})\right\} = \mathbb{R}_{>0}\right\}$$

Too little

There is ρ s.t. $\mu(x : 1 \in \mathcal{R}_\rho(x)) > 1 - \varepsilon$

so flow along till Doubling Radius matches H-pair annulus

For most x, find t such that $\mathcal{R}_\rho(a_t.x) \cap H_{\rho,\rho^{-1},d(x,x')^{1/4}}(a_t.x, a_t.x')$ non-empty.

a_t-invariance and Weak Doubling

Apply a_t flow to change doubling radius

Since $[a_t \mu_x^{G_a^-} a_{-t}] = [\mu_{a_t x}^{G_a^-}]$ and $a_t \mu_x^{G_a^-}(B_r^{G_a^-}) a_{-t} = \mu_x^{G_a^-}(B_{e^{2t} r}^{G_a^-})$

$$e^{2t} r \in \mathcal{R}_\rho(x) \text{ iff } r \in \mathcal{R}_\rho(a_t.x).$$

Therefore

$$\mathcal{R}_\rho(x) = \left\{ e^{2t} : a_t.x \in X_{\rho,1\text{-Doubling}} = \{x : 1 \in \mathcal{R}_\rho(x)\} \right\}.$$

Maximal Ergodic Theorem

Maximal Ergodic Theorem for a_t-action

For any $\mu(Y) > 1 - \delta$,

$$X_{a\text{-Erg}(Y)} = \left\{ x : \frac{1}{\tau} \int_0^\tau 1_Y(a_t.x)dt \geq 1 - \delta^{1/2} \right\}$$

$$\mu(X_{a\text{-Erg}(Y)}) \geq 1 - c\delta^{1/2}.$$

Apply to $Y = X_{u\text{-Erg}}(X_1) \cap X_{\rho,1\text{-Doubling}}$, $\delta = \varepsilon^{1/2}$

Take compact subset

$$X_{\text{Erg}} \subset X_{a\text{-Erg}(X_{u\text{-Erg}}(X_1) \cap X_{\rho,1\text{-Doubling}})} \cap X_{a^{-1}\text{-Erg}(X_{u\text{-Erg}}(X_1) \cap X_{\rho,1\text{-Doubling}})}$$

of mass $\mu(X_{\text{Erg}}) \geq 1 - \varepsilon^{1/4}$.

Preparation: Commutators of opposite horospherical

Goal: Upgrade H-property to hold simultaneously along part of A-orbit.

Lemma

For any $\delta < 1 < s < 1/\delta$

$$u_s n_\delta \in B_{s\delta}^{\mathrm{SL}_2(\mathbb{R})} u_{\frac{s}{1+\delta s}}.$$

Proof.

$$u_s n_\delta = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} = \begin{bmatrix} 1+s\delta & s \\ \delta & 1 \end{bmatrix} = \begin{bmatrix} 1+s\delta & 0 \\ \delta & \frac{1}{1+s\delta} \end{bmatrix} \begin{bmatrix} 1 & \frac{s}{1+s\delta} \\ 0 & 1 \end{bmatrix} \in B_{s\delta}^{\mathrm{SL}_2(\mathbb{R})} u_{\frac{s}{1+\delta s}}$$

□

Construction of H-pairs

Lemma (Uniform H-property along A-orbit)

Fix $\rho \in (0,1)$, let $\delta \ll 1$, $x, x' \in X_1 \cap B_\delta(x) \setminus B_1^{G_a^-}.x$.
Write

$$x = gt.x', \quad g = a_{s_a} u_{s_-} n_{s_+} \in AG_a^- G_a^+$$

Case 1: $\xi_1 := |s_a|^{-1} < |s_+|^{-10/21}$

$$\xi_1 \in H_{\rho,\rho^{-1},c\delta}(a_t.x, a_t.x') \quad \textit{for all } t \in [0, 0.01 \log \xi_1]$$

Uniform H-property along A-orbit

$$x, x' = gt.x \in X_1 \cap B_\delta(x) \setminus B_1^{G_a^-}.x \qquad \|g\|, \|t\| < \delta$$

Proof.

Decompose: $g = aun$, $a = a_{s_a}$, $u = u_{s_-}$, $n = n_{s_+}$ such that $|s_a|,|s_-|,|s_+| < \delta$

G isometry on $T.x$: $d(u_\xi a_t.x', u_\xi a_t g.x) < \delta$ for $|\xi| > 1, t > 0$

Rearrange
$$u_\xi a_t a_{s_a} u_{s_-} n_{s_+}.x = u_\xi a_{s_a} u_{e^{-2t}s_-} n_{e^{2t}s_+}.a_t x$$
$$= a_{s_a} u_{e^{+2s_a}\xi + e^{-2t}s_-} n_{e^{2t}s_+}.a_t x$$

Lemma
$$\in a_{s_a} B^{\mathrm{SL}_2(\mathbb{R})}_{2\xi e^{2t}|s_+|} u_{\frac{e^{+2s_a}\xi + e^{-2t}s_-}{1+e^{2t}s_+(e^{+2s_a}\xi + e^{-2t}s_-)}} a_t.x$$

Taylor
$$\subset B^{\mathrm{SL}_2(\mathbb{R})}_R u_{\xi + 2s_a\xi - e^{2t}s_+\xi^2} a_t.x$$

assuming $|\xi^2 e^{2t} s_+|, |2\xi s_a| \leq 1$ and

$$R = 2\max\left(2\xi e^{2t}|s_+|, e^{-2t}|s_-|, |\xi|^{-1}\right)$$

□

Uniform H-property along A-orbit

Proof.

Details for Taylor step:

$$a_{s_a} B^{\mathrm{SL}_2(\mathbb{R})}_{2\xi e^{2t}|s_+|} u_{\frac{e^{+2s_a}\xi + e^{-2t}s_-}{1+e^{2t}s_+(e^{+2s_a}\xi+e^{-2t}s_-)}} \subset B^{\mathrm{SL}_2(\mathbb{R})}_{20R} u_{\xi+2s_a\xi - e^{2t}s_+\xi^2}$$

assuming $|\xi^2 e^{2t}s_+|, |2\xi s_a| \leq 1$ and

$$R = \max\left(2\xi e^{2t}|s_+|, e^{-2t}|s_-|, |\xi|^{-1}\right)$$

a_{s_a} absorbed by $|s_a| \leq |\xi|^{-1}$

$$e^{+2s_a}\xi + e^{-2t}s_- = \xi + 2s_a\xi + e^{-2t}s_- + \mathcal{O}(\xi^{-1})$$

$$(1 + e^{2t}s_+(e^{+2s_a}\xi + e^{-2t}s_-))^{-1} = 1 - e^{2t}s_+\xi + \mathcal{O}(e^{2t}s_+s_a\xi + s_-s_+ + (e^{2t}s_+\xi)^2)$$

□

Uniform H-property along A-orbit

Proof.

Details for Taylor step:

$$a_{s_a} B^{\mathrm{SL}_2(\mathbb{R})}_{2\xi e^{2t}|s_+|} u_{\frac{e^{+2s_a}\xi + e^{-2t}s_-}{1+e^{2t}s_+(e^{+2s_a}\xi+e^{-2t}s_-)}} \subset B^{\mathrm{SL}_2(\mathbb{R})}_{20R} u_{\xi+2s_a\xi - e^{2t}s_+\xi^2}$$

$$\text{assuming } |\xi^2 e^{2t} s_+|, |2\xi s_a| \leq 1 \text{ and}$$

$$R = \max\left(2\xi e^{2t}|s_+|, e^{-2t}|s_-|, |\xi|^{-1}\right)$$

$$(\xi + 2s_a\xi + e^{-2t}s_- + \mathcal{O}(\xi^{-1}))$$
$$(1 - e^{2t}s_+\xi + \mathcal{O}(e^{2t}s_+s_a\xi + s_-s_+ + (e^{2t}s_+\xi)^2))$$
$$= (\xi + 2s_a\xi + R + R) + (-e^{2t}s_+\xi^2 + R + R + R)$$
$$+\mathcal{O}\left((R|\xi s_a| + R) + (R) + (1^2|\xi|^{-1})\right)$$

□

Proof of Case 1: $\xi_1 := \frac{1}{2}|s_a|^{-1} < \frac{1}{2}|s_+|^{-10/21}$.

$$\text{Summary: } u_\xi a_t a_{s_a} u_{s_-} n_{s_+}.x \in B_{20R}^{\mathrm{SL}_2(\mathbb{R})} u_{\xi+2s_a\xi-e^{2t}s_+\xi^2}$$

$$\text{if } |\xi^2 e^{2t} s_+|, |2\xi s_a| \le 1 \text{ and } R = \max\left(2\xi e^{2t}|s_+|, e^{-2t}|s_-|, |\xi|^{-1}\right)$$

$$\text{Assume } \xi \in [\rho\xi_1, \xi_1], \quad t \in [0, 0.01 \log \xi_1]$$

Then $|\xi^2 e^{2t} s_+| \le \xi^{2.02}|s_+| \le |s_+|^{-20.2/21+1} = |s_+|^{0.08} \le \delta^{0.08}$
and $R \le c\delta$.
Also $|\xi + 2s_a\xi - e^{2t}s_+\xi^2 - \xi| \in [\rho, 2]$. Hence

$$\xi_1 \in \mathrm{H}_{\rho,\rho^{-1},20c\delta}(a_t.x, a_t.x') \quad \text{for all } t \in [0, 0.01 \log \xi_1]$$

□

Construction of H-pairs with Doubling Condition

Lemma (Flow to Doubling H-pairs)

Let $x, x' \in X_{Erg} \cap T.x \cap B_\delta(x) \setminus B_1^{G_a^-}.x$.
There is $t \geq 0$ *such that*

$$a_t.x, a_t.x' \in X_{u\text{-}Erg}$$

and

$$H_{\rho,\rho^{-1},\delta^{1/4}}(a_t.x, a_t.x') \cap \mathcal{R}_\rho(a_t.x) \neq \emptyset.$$

This gives the necessary assumptions for the "Leafwise Measure Lemma" and thereby completes the proof of the theorem.

Construction of H-pairs with Doubling Condition

Lemma (Flow to Doubling H-pairs.)

$x, x' \in X_{Erg} \cap B_\delta(x)$

$$H_{\rho,\rho^{-1},\delta^{1/4}}(a_t.x, a_t.x') \cap \mathcal{R}_\rho(a_t.x) \neq \emptyset \qquad a_t.x, a_t.x' \in X_{u\text{-}Erg}$$

Proof for Case 1.

Lemma $\xi_1 \in \mathsf{H}_{\rho,\rho^{-1},c\delta}(a_t.x, a_t.x')$ for all $t \in [0, 0.01 \log \xi_1]$

Recall $e^{2\tau} r \in \mathcal{R}_\rho(z)$ iff $r \in \mathcal{R}_\rho(a_\tau.z)$.
Apply to $z = a_t.x, r = \xi_1 e^{-2\tau}, \tau = \frac{1}{2} \log \xi_1$.

Remains to show

$$a_{t+\frac{1}{2}\log \xi_1}.x \in X_{\rho,\text{1-Doubling}} = \{x : 1 \in \mathcal{R}_\rho(x)\}$$

□

Construction of H-pairs with Doubling Condition

Lemma (Flow to Doubling H-pairs.)

$x, x' \in X_{Erg} \cap B_\delta(x)$. *There is* $t \in [0, 0.01 \log \xi_1]$ *such that*

$$a_{t+\frac{1}{2}\log \xi_1}.x \in X_{\rho,1\text{-}Doubling} = \{x : 1 \in \mathcal{R}_\rho(x)\}, \; a_t.x, a_t.x' \in X_{u\text{-}Erg}$$

Proof for Case 1.

Definition $x, x' \in X_{\mathrm{Erg}} \subset$

$$X_{a\text{-}\mathrm{Erg}(X_{u\text{-}\mathrm{Erg}} \cap X_{\rho,1\text{-}\mathrm{Doubling}})} \cap X_{a^{-1}\text{-}\mathrm{Erg}(X_{u\text{-}\mathrm{Erg}} \cap X_{\rho,1\text{-}\mathrm{Doubling}})}$$

Erg. Thm

$$\int_0^{0.01 \log \xi_1} 1_{X_{u\text{-}\mathrm{Erg}}}(a_t.x) 1_{X_{u\text{-}\mathrm{Erg}}}(a_t.x') dt$$

$$\geq (1 - 2\varepsilon^{1/4}) 0.01 \log \xi_1$$

□

Proof for Case 1.

$x, x' \in X_{\text{Erg}} \subset X_{a\text{-Erg}(X_{u\text{-Erg}} \cap X_{\rho,1\text{-Doubling}})} \cap X_{a^{-1}\text{-Erg}(X_{u\text{-Erg}} \cap X_{\rho,1\text{-Doubling}})}$

$$\int_0^{0.01 \log \xi_1} 1_{X_{u\text{-Erg}}}(a_t.x) 1_{X_{u\text{-Erg}}}(a_t.x') dt$$

$$\geq (1 - 2\varepsilon^{1/4}) 0.01 \log \xi_1$$

Similarly,

$$\int_0^{\frac{1}{2}\log \xi_1 + 0.01 \log \xi_1} 1_{X \setminus X_{\rho,1\text{-Doubling}}}(a_t.x) dt$$

$$\leq \varepsilon^{1/4} \log \xi_1$$

and in particular (by restriction)

$$\int_0^{0.01 \log \xi_1} 1_{X \setminus X_{\rho,1\text{-Doubling}}}(a_{t+\frac{1}{2}\log \xi_1}.x) dt \leq \varepsilon^{1/4} \log \xi_1$$

□

Construction of H-pairs with Doubling Condition

Lemma (Flow to Doubling H-pairs.)

$x, x' \in X_{Erg} \cap B_\delta(x)$. *There is* $t \in [0, 0.01 \log \xi_1]$ *such that*

$a_{t+\frac{1}{2}\log \xi_1}.x \in X_{\rho,1\text{-}Doubling} = \{x : 1 \in \mathcal{R}_\rho(x)\}$ *and* $a_t.x, a_t.x' \in X_{u\text{-}Erg}$

Proof for Case 1.

$$\int_0^{0.01\log\xi_1} 1_{X_{u\text{-Erg}}}(a_t.x) 1_{X_{u\text{-Erg}}}(a_t.x')dt$$

$$\geq (1 - 2\varepsilon^{1/4}) 0.01 \log \xi_1$$

$$\int_0^{0.01\log\xi_1} 1_{X \setminus X_{\rho,1\text{-Doubling}}}(a_{t+\frac{1}{2}\log\xi_1}.x)dt \leq \varepsilon^{1/4}\frac{1}{2}\log\xi_1$$

so for $\varepsilon^{1/4}(0.02 + 1) < 0.01$, there is $t \in [0, 0.01\xi_1]$ such that $a_{t+\frac{1}{2}\log\xi_1}.x \in X_{\rho,1\text{-Doubling}}$ and $a_t x, a_t x' \in X_{u\text{-Erg}}$.

□

Construction of H-pairs

Lemma (Uniform H-property along A-orbit)

Fix $\rho \in (0,1)$*, let* $\delta \ll 1$*,* $x, x' \in X_1 \cap B_\delta(x) \setminus B_1^{G_a^-}.x$*.*
Write

$$x = gt.x', \quad g = a_{s_a} u_{s_-} n_{s_+} \in A G_a^- G_a^+$$

Case 1: $\xi_1 := |s_a|^{-1} < |s_+|^{-10/21}$

$$\xi_1 \in H_{\rho,\rho^{-1},c\delta}(a_t.x, a_t.x') \quad \textit{for all } t \in [0, 0.01 \log \xi_1]$$

Case 2: Otherwise $\xi_1 := |s_+|^{-1/2}$

$$\xi_1 e^{-t} \in H_{\rho,2\rho^{-1},c\delta^{0.495}}(a_t.x, a_t.x') \quad \textit{for all } t \in [0.05 \log \xi_1, 0.1 \log \xi_1]$$

Proof of Case 2: $\xi_1 := |s_+|^{-1/2}$, $|s_a|^{-1} > |s_+|^{-10/21}$.

$$\text{Summary: } u_\xi a_t a_{s_a} u_{s_-} n_{s_+}.x \in B_{20R}^{\mathrm{SL}_2(\mathbb{R})} u_{\xi+2s_a\xi-e^{2t}s_+\xi^2}$$

$$\text{if } |\xi^2 e^{2t} s_+|, |2\xi s_a| \leq 1 \text{ and } R = \max\left(2\xi e^{2t}|s_+|, e^{-2t}|s_-|, |\xi|^{-1}\right)$$

$$\text{Assume } \xi \in [\rho\xi_1 e^{-t}, \xi_1 e^{-t}], \quad t \in [0.05 \log \xi_1, 0.1 \log \xi_1]$$

Then $|\xi^2 e^{2t} s_+| \leq \xi_1^{0.95}|s_+| \leq |s_+|^{0.475-10/12} \leq \delta^{0.001}$
and
$R = \max\left(\xi e^{2t}|s_+|, e^{-2t}|s_-|, \xi^{-1}\right) \leq \max\left(\delta^{0.495}, \delta, \delta^{1/2}\right) = \delta^{0.495}$.
Also $|\xi + 2s_a\xi - e^{2t}s_+\xi^2 - \xi| \in [\rho/2, 1]$. Hence

$e^{-t}\xi_1 \in \mathrm{H}_{\rho, 2\rho^{-1}, 20\delta^{0.495}}(a_t.x, a_t.x')$ for all $t \in [0.05 \log \xi_1, 0.1 \log \xi_1]$.

□

Construction of H-pairs with Doubling Condition

Lemma (Flow to Doubling H-pairs.)

$x, x' \in X_{Erg} \cap B_\delta(x)$

$$H_{\rho,\rho^{-1},\delta^{1/4}}(a_t.x, a_t.x') \cap \mathcal{R}_\rho(a_t.x) \neq \emptyset \qquad a_t.x, a_t.x' \in X_{u\text{-}Erg}$$

Proof for Case 2.

Lemma $\xi_1 e^{-t} \in \mathrm{H}_{\rho,2\rho^{-1},20\delta^{0.495}}(a_t.x, a_t.x'),\ t \in [0.05 \log \xi_1, 0.1 \log \xi_1]$

Recall $e^{2\tau} r \in \mathcal{R}_\rho(z)$ iff $r \in \mathcal{R}_\rho(a_\tau.z)$.
Apply to $z = a_t.x, r = \xi_1 e^{-t-2\tau}, \tau = \frac{1}{2}\log\xi - \frac{1}{2}t$.

Remains to show

$$a_{\frac{1}{2}t+\frac{1}{2}\log\xi_1}.x \in X_{\rho,\text{1-Doubling}} = \{x : 1 \in \mathcal{R}_\rho(x)\}$$

□

Construction of H-pairs with Doubling Condition

Lemma (Flow to Doubling H-pairs.)

$x, x' \in X_{Erg} \cap B_\delta(x)$. *There is* $t \in [0.05 \log \xi_1, 0.1 \log \xi_1]$ *such that*

$a_{\frac{1}{2}t+\frac{1}{2}\log \xi_1}.x \in X_{\rho,1\text{-}Doubling} = \{x : 1 \in \mathcal{R}_\rho(x)\},\ a_t.x, a_t.x' \in X_{u\text{-}Erg}.$

Proof for Case 2.

$\int_{0.05\log\xi_1}^{0.1\log\xi_1} 1_{X_{u\text{-Erg}}}(a_t.x) 1_{X_{u\text{-Erg}}}(a_t.x')dt \geq (1-4\varepsilon^{1/4})0.05\log\xi_1$

By restricting again from $\int_0^{\log\xi_1}$ to $\int_{\frac{1}{2}\log\xi_1+\frac{1}{2}0.05\log\xi_1}^{\frac{1}{2}\log\xi_1+\frac{1}{2}0.1\log\xi_1}$

$$\int_{0.05\log\xi_1}^{0.1\log\xi_1} 1_{X\setminus X_{\rho,1\text{-Doubling}}}(a_{\frac{1}{2}t+\frac{1}{2}\log\xi_1}.x)dt \leq \varepsilon^{1/4}\log\xi_1$$

The lemma follows for sufficiently small ε. □