מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

Effective Intrinsic Ergodicity For Countable Markov Shifts

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Structure for today's Ergodic Seminar

1. Intrinsic Ergodicity
2. Effective Intrinsic Ergodicity
3. Spectral Gap
4. Escape of Mass
5. Effective Intrinsic Ergodicity for countable TMS
6. Pressure
7. Restricted Entropy
8. Sharp Effective Intrinsic Ergodicity

Intrinsic Ergodicity

Parry Measure

Let $\mathscr{G}$ be a **finite directed graph** with primitive adjacency matrix A.

Perron-Frobenius Theorem: There exist positive left and right eigenvectors ℓ and r to a unique maximal eigenvalue λ s.t. $\lambda = \ell A r$.

$$P_{ij} = \frac{A_{ij} r_j}{\lambda r_i}, \quad p_i = \ell_i r_i$$

define a stochastic matrix with equilibrium probability p, $pP = p$.

Let (Σ^+, σ) be the associated **shift space of one-sided infinite paths** on $\mathscr{G}$.

The topological entropy is $h_{\text{top}}(\sigma) = \log \lambda$.

Theorem (Parry '64)

The unique measure of maximal entropy of (Σ^+, σ) is

$$\mu_{p,P}([x_0, \ldots x_{n-1}]) = p_{x_0} \prod_{i=0}^{n-2} P_{x_i x_{i+1}}, \quad h_{\mu_{p,P}}(\sigma) = h_{\text{top}}.$$

Intrinsically Ergodic

Definition (Weiss '70)

Call a dynamical system intrinsically ergodic if there is a unique measure of maximal entropy.

Gurevich Entropy

Let $\mathscr{G}$ be a strongly connected directed **countable graph**, and let (Σ^+, σ) be the *non-compact* associated shift space of infinite one-sided paths on $\mathscr{G}$.

Let $Z_n(a) = \#\{\underline{x} \in \Sigma^+ : x_0 = a, \sigma^n(\underline{x}) = \underline{x}\}$ for a fixed state $a \in V(\mathscr{G})$.

Definition (Gurevich '69)

$$h_G = \lim_{n \to \infty} \frac{1}{n} \log Z_n(a)$$

Theorem (Gurevich '70)

Suppose $h_G < \infty$. Then (Σ^+, σ) is positively recurrent if and only if it is intrinsically ergodic.

Let $Z_n^*(a) = \#\{\underline{x} \in \Sigma^+ : x_0 = a, \sigma^n(\underline{x}) = \underline{x}, x_i \neq a \ (0 < i < n)\}$. Σ^+ is **positively recurrent** if $\sum \lambda^{-n} Z_n(a) = \infty$ and $\sum n \lambda^{-n} Z^*(a) < \infty$ where $\lambda := e^{h_G}$. (Equivalently, with respect to the "Parry measure", the Markov chain returns to a almost surely with finite expected first return time).

Effective Intrinsic Ergodicity

Einsiedler Inequality

Let $T : \mathbb{T} \to \mathbb{T}$, $x \mapsto 2x$. Let m be the Lebesgue measure (= unique measure of maximal entropy).

Theorem (Polo '11)[1]

For any T-invariant probability μ on $\mathbb{T}$ and any Lipschitz function f,

$$\left| \int f dm - \int f d\mu \right| \ll \|f\|_{\mathrm{Lip}} \left(h_m(T) - h_\mu(T) \right)^{\frac{1}{3}} .$$

The implicit constant is independent of μ and f.

Polo's Thesis '11: Also Cat Map case $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right] : \mathbb{T}^2 \to \mathbb{T}^2$.

[1] Attributes proof to Einsiedler

Kadyrov's Theorem

Let (Σ^+, σ) be a topologically mixing subshift of finite type. Let m be the Parry measure (= unique measure of maximal entropy). Let $\mathcal{H}_\beta$ be the space of Hölder continuous functions with norm $\|f\|_\beta := \|f\|_\infty + \sup_{\underline{x}\neq\underline{y}} \frac{|f(\underline{x})-f(\underline{y})|}{d(\underline{x},\underline{y})^\beta}$.

Theorem (Kadyrov '14)

For any σ-invariant probability μ on Σ^+ and any $f \in \mathcal{H}_\beta$,

$$\left| \int f dm - \int f d\mu \right| \ll \|f\|_\beta \left(h_m(\sigma) - h_\mu(\sigma)\right)^{\frac{1}{2}} .$$

The implicit constant is independent of μ and f.

Proof hinges on the following two facts:

- Information function of m is cohomologous to a constant via integrable transfer function: $-\log \mathbb{E}_m(1_{[x_0]}|\sigma^{-1}\mathcal{B}_{\Sigma^+})(\underline{x}) = -\log \frac{\ell_{x_0}}{\lambda \ell_{x_1}}$
- $\|T^n f - m(f)\|_\infty \ll \|f\|_\beta \kappa^{-n}$ where T is the transfer operator satisfying $Tf \circ \sigma = \mathbb{E}_m(f|\sigma^{-1}\mathcal{B}_{\Sigma^+})$

Mind the gap!

Spectral Gap

Countable Alphabet Troubles

Spectral gap
$\|T^n f - m(f)\|_\infty \ll \|f\|_\beta \kappa^{-n}$

- $Tf = \lambda^{-1} M_h^{-1} L M_h$ (transfer operator) where M_h is the multiplication operator by h, the eigenfunction of Ruelle's operator L ($Lf(\underline{x}) = \sum_{\sigma(\underline{y})=\underline{x}} f(\underline{y})$).
- This cannot possibly hold for general infinite Markov shifts. (Apply to f compactly supported and Σ^+ locally compact).

Strongly Positively Recurrence

Recall that $Z_n^*(a)$ is the number of first return loops at a of length n.

Definition

Let

$$h_G^*(a) = \lim_n \frac{1}{n} \log Z_n^*(a).$$

Then (Σ^+, σ) is called Strongly Positively Recurrent (SPR) if

$$h^*(a) < h_G < \infty.$$

Theorem (Cyr-Sarig '08)

(Σ^+, σ) is SPR **iff** it has the Spectral Gap Property.

Spectral Gap Property

Ruelle's operator: For a function $\psi : \Sigma^+ \to \mathbb{R}$ let

$$L_\psi f(\underline{x}) = \sum_{\sigma(\underline{y})=\underline{x}} e^{\psi(\underline{y})} f(\underline{y}).$$

Definition

The Spectral Gap Property holds if $L = L_0$ acts quasi-compactly on a *nice* Banach space, and if

$$z \mapsto L_{z\psi}$$

is an analytic family of operators for any $\psi \in \mathcal{H}_\beta$.

Let’s get lost for a moment.

Escape of Mass

Escape of Mass

Assume $\mathscr{G}$ strongly connected with finite Gurevich entropy.

Theorem (Ruette '03)

If (Σ^+, σ) is not SPR then there exists a sequence of asymptotic equilibrium measures μ_n escaping to infinity:

$$h_{\mu_n} \to h_G \text{ but } \mu_n([a]) \to 0 \quad \forall a \in V(\mathscr{G})$$

Related criterion by Gurevich-Zargaryan '88.

Corollary

If (Σ^+, σ) is not SPR then there cannot be any effective intrinsic ergodicity.

Effective Intrinsic Ergodicity for countable TMS

Effective Intrinsic Ergodicity for countable TMS

Assume that (Σ^+, σ) is topologically mixing, $h_G < \infty$. Recall:

Theorem (Gurevich '70)

(Σ^+, σ) is positively recurrent **iff** it is intrinsically ergodic.

Theorem (Ruette '03, R-Sarig '21)

(Σ^+, σ) is SPR **iff** it is effectively intrinsically ergodic:

$\forall \psi \in \mathcal{H}_\beta$, $\forall \mu$ σ-invariant probabilities

$$|m(\psi) - \mu(\psi)| \ll \|\psi\|_\beta \left(h_m(\sigma) - h_\mu(\sigma)\right)^{\frac{1}{2}}$$

where m is the unique measure of maximal entropy.

Use the Pressure.

Pressure

Pressure function

For $\psi \in \mathcal{H}_\beta$ let $\psi_n(\underline{x}) = \sum_{i=0}^{n-1} \psi(\sigma^i(\underline{x}))$ and define

$$Z_n(\psi, a) := \sum_{\sigma^n(\underline{x})=\underline{x}} e^{\psi_n(\underline{x})} 1_{[a]}(\underline{x})$$

Suppose (Σ^+, σ) is topologically mixing. The Gurevich-Sarig pressure is

$$P_{\mathrm{GS}}(\psi) = \lim_n \frac{1}{n} \log Z_n(\psi, a)$$

and satisfies, for every $\underline{x} \in \Sigma^+$,

$$P_{\mathrm{GS}}(\psi) = \lim_n L_\psi^n 1_{[a]}(\underline{x}).$$

Using the Spectral Gap Property, $t \mapsto P_{\mathrm{GS}}(t\psi)$ is analytic around 0:

Taylor Expansion

$$P_{\mathrm{GS}}(t\psi) = h_G + m(\psi)t + \frac{1}{2}\sigma_m^2(\psi)t^2 + \dots$$

Asymptotic Variance, Variational Principle

Taylor Expansion

$$P_{GS}(t\psi) = h_G + m(\psi)t + \frac{1}{2}\sigma_m^2(\psi)t^2 + \dots$$

where

- $\sigma_m^2(\psi) = \lim_n \frac{1}{n}\mathrm{Var}_m[\psi_n]$ the asymptotic variance where $\psi_n = \sum_{k=0}^{n-1} \psi \circ \sigma^k$.

Variational Principle (Gurevich '69, Sarig '99)

$$P_{GS}(\psi) = \sup_{\mu} \left(h_\mu(\sigma) + \mu(\psi)\right)$$

- For t small $m_{t\psi}$, *the* equilibrium measure for the potential $t\psi$ exists, i.e. $P_{GS}(t\psi) = h_{m_{t\psi}}(\sigma) + m_{t\psi}(t\psi)$. (Uniqueness by Buzzi-Sarig, existence by Cyr-Sarig.)

Please restrict your measure.

Restricted Entropy

Restricted Entropy function

Restrict to hardest case:

Instead of fixing μ in $|m(\psi) - \mu(\psi)| \leq |h_m(\sigma) - h_\mu(\sigma)|^\alpha$

consider all ν with $\nu(\psi) = \mu(\psi)$ and minimize RHS.

Definition (Restricted entropy)

$$H_\psi(a) = \sup\{h_\nu(\sigma) : \nu(\psi) = a\}$$

- $H_\psi(m(\psi)) = h_m(\sigma)$.
- $\frac{d}{ds}P_{GS}(s\psi)\big|_{s=t} = m_{t\psi}(\psi)$ analytic if t close to zero (by SPR).
- Suppose a close to $m(\psi)$. Then $\exists t \in \mathbb{R}$ s.t. $m_{t\psi}(\psi) = a$.
- Hence $H_\psi(a)$ attained by $m_{t\psi}$: $H_\psi(a) = P_{GS}(t\psi) - ta$
- This implies that $-H_{\boldsymbol{\Psi}}(a)$ is the Legendre transform of $P_{GS}(t\boldsymbol{\Psi})$, so can use known formulas for derivatives of $H_{\boldsymbol{\Psi}}(a)$.

Lemma (Taylor approximation) Assuming $|m(\psi) - \mu(\psi)|$ small depending on ε and ψ

$$H_\psi(\mu(\psi)) = H_\psi(m(\psi)) - e^{\pm\varepsilon}\frac{1}{2\sigma_m^2(\psi)}(m(\psi) - \mu(\psi))^2.$$

Sharp Effective Intrinsic Ergodicity

O. Sarig's version of Effective Intrinsic Ergodicity with optimal constant and exponent

Lemma (Assuming $|m(\psi) - \mu(\psi)|$ small depending on ε and ψ)

$H_\psi(\mu(\psi)) = H_\psi(m(\psi)) - e^{\pm\varepsilon} \frac{1}{2\sigma_m^2(\psi)} (m(\psi) - \mu(\psi))^2.$

- Remark: $\sigma_m(\psi) = 0$ iff ψ is cohomologous to a constant.
- By definition of H_ψ, $H_\psi(m(\psi)) - H_\psi(\mu(\psi)) \le h_m - h_\mu$, so
- $|h_m - h_\mu|$ small $\rightsquigarrow$ $|m(\psi) - \mu(\psi)|$ small if $\|\psi\|_\beta \ll 1$. (Does **not** require $|m(\psi) - \mu(\psi)|$ to be small.)

Corollary (Assume $h_m \approx h_\mu$, $\|\psi\|_\beta \le 1$)

$$|m(\psi) - \mu(\psi)| \le e^{\varepsilon} \sqrt{2} \sigma_m(\psi) \left(h_m(\sigma) - h_\mu(\sigma)\right)^{\frac{1}{2}}$$

Concavity of Restricted Entropy Function

Lemma (Taylor approximation) Assuming $|m(\psi) - \mu(\psi)|$ small depending on ε and ψ

$H_\psi(\mu(\psi)) = H_\psi(m(\psi)) - e^{\pm\varepsilon} \frac{1}{2\sigma_m^2(\psi)} (m(\psi) - \mu(\psi))^2.$

$|h_m - h_\mu|$ small $\rightsquigarrow$ $|m(\psi) - \mu(\psi)|$ small if $\|\psi\|_\beta \ll 1$. (Does **not** require $|m(\psi) - \mu(\psi)|$ to be small.)

Lemma (Concavity)

$$|m(\psi) - \mu(\psi)| \ll \sigma_m(\psi)^{-2} (H_\psi(m(\psi)) - H_\psi(\mu(\psi)))$$

Concavity of Restricted Entropy Function

Lemma (Concavity)

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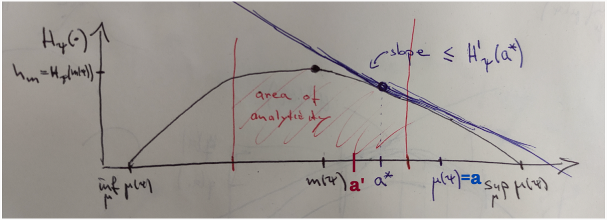

- Suppose $m(\psi) < a^* < a = \mu(\psi)$ where H_ψ still analytic at a^*.
- Concavity: $H_\psi(a) - H_\psi(a^*) \leq (a - a^*)H'_\psi(a^*)$
- Mean value theorem: $H'_\psi(a^*) = H'_\psi(m(\psi)) + H''_\psi(a')(a^* - m(\psi))$
- Global maximum, concavity: $H'_\psi(a^*) \leq H''_\psi(m(\psi))(a^* - m(\psi))$
- Hence $H_\psi(a) - H_\psi(a^*) \leq -(2\sigma_m(\psi))^{-2}(a^* - m(\psi))(a - a^*)$
- $a^x - m(\psi)$ can be bounded from below ($\gg \sigma_m(\psi)^4$)

General Measures

Theorem (Effective Intrinsic Ergodicity)

$$|m(\psi) - \mu(\psi)| \ll \|\psi\|_\beta (h_m - h_\mu)^{\frac{1}{2}}$$

- Follows from $\sigma_m(\psi) \ll \|\psi\|_\beta$ for large entropy μ.
- Small entropy μ needs a reverse bound to apply concavity lemma.
- Or rather, there is ψ_μ that satisfies the reverse bound and gives same integrals to m and μ as ψ does, and $\|\psi_\mu\|_\beta \ll \|\psi\|_\beta$.
- Both bounds use SPR.