A local criterion for regularity of a system of points * †

B. N. Delone, N. P. Dolbilin, M. I. Shtogrin, R. V. Galiulin

Any Crystal structure (i.e., a structure of an infinite ideal mono-crystal) is regular. The word "regular" means that the complete group of motions of the space that superpose this structure with itself is discrete (which is evident in view of the atomic structure) and has finite fundamental domain. This last circumstance should have been considered the main principle of crystalography, since it lies in the foundation of the geometric crystalography, while its physical cause haven't been discovered yet.

In general, a regular system of points (an orbit) is a set of points obtained from some point of the space by means of some subgroup of the space motions. But in crystalography, as we have just mentioned, discrete groups G and finite fundamental domains have a special role. Therefore, later on by regular system we mean only orbits with respect to such groups. Any crystal structure obviously consists of a finite amount of regular systems of atoms, related to its group G.

A regular system of points, in view of the discreteness of G and the finiteness of its fundamental domain, is an (r, R)-system (r and R are some positive real numbers). Let us remind [1] that an (r, R)-system is a set of points of the space that satisfies the following conditions:

1. The distance between any two distinct points of the set is not less than r;
2. In the interior and on the boundary of any ball of radius R, wherever its center is, there is at least one point from the set.

For any given (r, R)-system, there exists a maximal number that satisfies condition (1) and a minimal number that satisfies the condition (2). Below, each time talking about an r, R-system, we will mean that r and R are these extremal values of the given system. Of course, not every (r, R)-system is regular. In order for (r, R) to be regular, it is necessary and sufficient, as Fyodorov said, that from each point of the system all the other points would "look the same" as from any other point. The same was more rigorously

*Published in *Dokl. Akad. Nauk SSSR, 1976, Volume 227, Number 1, 19-21.*

†Translated from Russian by Karina Samvelyan, 24.09.2019.

[1] A reference to Delone, 1937

formulated by Hilbert and Con-Fossen. Let us take an arbitrary point of the system and connect it to all the other points by segments. This way, for the given point we obtain an infinite "spider". The system will be regular if and only if for all the points these "spiders" and congruent. In other words, for any two points of a regular system there exists a motion that transfers the first point to the second one and superposes the whole system with itself.

The goal of the current work is to show that for all the three spaces (Euclidean, spherical, Lobachevsky space) of any dimension, for the regularity of an (r, R)-system it is enough to have congruence of certain not big finite, so called stable "spiders" of all of its points.

1 A stable spider of a point of an (r, R)-system

Let S be an (r, R)-system in the space E^n, where E^n is an n-dimensional that is either Euclidean, or Lobachevsky, or spherical.

Consider a ball of an arbitrary radius ρ centered at a point $A \in S$. The set of points lying inside or on the boundary of this ball would be denoted by $S_A(\rho)$, and the group of rotations of E^n around the point A that superpose $S_A(\rho)$ with itself would be denoted by $H_A(\rho)$.

For any ρ, the set $S_A(\rho)$ is finite, in view of the r-property of S. Let us show that for $\rho = 2R$ the set $S_A(2R)$ is n-dimensional *affine independent* [2], from which it will follow that the group $H_A(2R)$ is finite.

Indeed, any point of S that is situated inside or on the boundary of some ball of radius R, on the boundary of which lies A, belongs to $S_A(2R)$. In view of the r-property and the R-property of the system, in or on the boundary of such a ball there is a point A_1, with A_1 different from A. This is because if no point of S except A was belonging to this ball, it would have been possible to increase the ball, first slightly moving it from the point A, such that the ball would not contain inside or on its boundary no points of S. And this contradicts condition (2). Analogously, inside or on the boundary of a ball of radius R that is tangent to the segment AA_1 at the point A, there exists a point A_2 that is different from A, such that the vector AA_2 is not co-linear to the vector AA_1. Inside or on the boundary of a ball of radius R that is tangent the two-dimensional plane AA_1A_2 at the point A, there exists a point A_3, that is different from the point A and such that the vector AA_3 is not co-planar to AA_1A_2, etc. This way we obtain an n-dimensional affine-independent set $A, A_1, A_2, \ldots A_n$, where all A_i belong to $S_A(2R)$.

Let us note that in a spherical space, the existence of the point A_{i+1}, for $0 \le i \le n-1$, $A_0 = A$, that lies outside of the plane $A_0A_1 \ldots A_i$, will be guaranteed for $R < c/4$, where c is the length of a big circle of the

[2][Transl.: The original term is *"n-dimensionally placed"*, also called *"reper"* in the proof. The set $A_0, A_1, \ldots, A_n$ is called (n-dimensional) affine independent if the set $\{(A_i - A_0)\}_i$ is linearly independent.]

sphere under consideration. Therefore, we will assume that in the case of a spherical space the inequality $R < c/4$ takes place.

Clearly, $H_A(\rho) \supseteq H_A(\rho')$ when $\rho < \rho'$. Since the group $H_A(2R)$ is finite and the system S is discrete, there exists a value ρ_0, such that for $\rho \geq \rho_0$ the group $H_A(\rho)$ is finite, and for $\rho < \rho_0$ the group $H_A(\rho)$ is infinite. Clearly, $r \leq \rho_0 < 2R$.

Let us show that for any point A of an (r, R)-system S, there exists a number ρ_1, bounded from above in terms of r, R and n, such that $H_A(\rho_1 + 2R) = H_A(\rho_1)$. Later on, by such ρ_1 for a point A we would mean the least such value. Indeed, denote the order of the group $H_A(\rho)$, which is finite in case $\rho \geq \rho_0$, by $h_A(\rho)$. The function $h_A(\rho)$ is defined for $\rho \geq \rho_0$ assumes natural values and does not increase. The graph of this function is clearly a step-graph, with the heights (above the ρ-axis) of the steps are natural numbers, and the height of each step, by Lagrange theorem on subgroups, is a divisor of the height of the preceding step. Therefore, the number of steps is finite, and in any case does not exceed $\nu + 1$, where ν is the number of prime divisors of $h_A(\rho_0)$. The last step is obviously infinite. As ρ_1 one should take the beginning of the first step whose length (along ρ) is greater than $2R$. The beginning, ρ_1, of such a step satisfies $\rho_1 < (\nu + 1)2R$, as it is easy to see.

The set $S_A(\rho_1 + 2R)$ will be called the stable set of the point A and we will denote it by S_A. The set $S_A(\rho_1)$ will be called the pre-stable set of A and we will denote it by $\tilde{S_A}$. The set of segments connecting the point A to all the points of S_A (or $\tilde{S_A}$) will be called the stable spider (or, respectively, pre-stable) or the point A, and will be denoted by P_A (or $\tilde{P_A}$). The theorem that we want to prove is as follows:

Theorem. *Let S be an (r, R)-system of points in the space E^n, where E^n is an n-dimensional space that is either Euclidean, spherical, or Lobachevsky. For S to be a regular system, it is sufficient for the stable spiders of all its points to be congruent, and then the system is uniquely defined by this spider.*

2 On $2R$-chains in an (r, R)-system

. A $2R$-chain in the system S is a sequence of points from S, such that the distance between any two adjacent points from this sequence does not exceed $2R$.

Lemma. *Any two points A and B from an (r, R)-system S can be connected by a $2R$-chain.*

Indeed, if the distance between A and B does not exceed $2R$, then A, B is such a $2R$-chain. If the distance between A and B exceeds $2R$, consider the ball of radius R that has A on its boundary and its center on the segment

AB (for a spherical space - the shorter AB). In view of the R-property, inside or on the boundary of this ball there lies a point A_1 of the system S, different from the point A, with $AA_1 \leq 2R$ and $A_1B < AB$. If $A_1B \leq 2R$, then A, A_1, B is the sought-for $2R$-chain. If $A_1B > 2R$, then consider the ball of radius $2R$, that has the point A_1 on its boundary and its center on the segment A_1B. Inside or on the boundary of this ball there is a point A_2 of S, different from the point A_1, where $A_1A_2 \leq 2R$ and $A_2B < A_1B$, and so on. This process, clearly, will terminate, since all the points $A_1, A_2, \dots$ obtained this way are distinct, lie inside a ball of radius AB centered at B, and there is a finite amount of such point from S, in view of the r-property of the system S.

3 Proof of the Theorem

Suppose that for the (r, R)-system the stable spiders of all its points are congruent.

Consider two points A and B of the system S and denote by ∂ a motion of the space E^n that transfers the stable spider P_A to the stable spider P_B. Let us prove that under the motion ∂, an arbitrary point C of S is moved to some point D of S. To this end, connect the points A and C by a $2R$-chain $AA_1A_2 \dots A_lC$. The point A_1 belongs to S_A and therefore under the motion ∂, the point A_1 is transfered to some point B_1 of S_B. Clearly, a ball of radius ρ_1 centered at A_1 lies inside a ball of radius $\rho_1 + 2R$ centered at the point A, while the ball of radius ρ_1 centered at B_1 lies inside the ball of radius $\rho_1 + 2R$ centered at B. Therefore, the pre-stable spider $\tilde{P_A}$ is transfered to the pre-stable spider $\tilde{P_B}$ under the motion ∂. But in this case, also the stable spider P_A is transfered to the stable spider P_B under the motion ∂, which follows from the congruence of the stable spiders and from the fact that every rotation of the pre-stable spider on itself, superposes also the stable spider of this point with itself.

Thus, the motion ∂, that transfered P_A into P_B also transfers P_{A_1} into P_{B_1}. Repeating the same considerations for the points A_1 and B_1, we will get convinced that the motion ∂ transfers the point $A_2 \in S_A$ into the point $B_2 \in S_B$, the pre-stable spider $\tilde{P_{A_2}}$ into the pre-stable spider $\tilde{P_{B_2}}$, and, consequently, the stable spider P_{A_2} into the stable spider P_{B_2}, and so on. Therefore, the motion ∂ transfers the point $C \in S$ into some point $D \in S$.

Since the motion ∂^{-1} transfers the stable spider P_B into the stable spider P_A, by analogous considerations, the motion ∂^{-1} transfers any point $D \in S$ into some point $C \in S$. In sum, we get that the considered motion ∂ superposes the (r, R)-system S with itself. Since A and B are arbitrary, the proof is complete.

Remark. The question whether for a system to be regular even smaller spiders are sufficient was not ascertained by us. But it has to be mentioned that the congruence of too small spiders is insufficient. For example[3], already on a two-dimensional sphere there is an (r, R)-system of points which is non-regular, despite all its $P_A(2R)$ spiders being congruent, although these spiders are in some sense not that small. Analogous examples exists also in the plane.

The authors think that the proven geometric theorem can be useful for finding out the physical reasons for regularity of the atomic structure a mono-crystal.

The Steklov Mathematical Institute of the Academy of Sciences, USSR, Moscow.
Submitted on the 10th of December, 1975.

[3]V. G. Ashkinuze, *Mathematical Enlightenment*, v. 1, 107 (1957).