Cut-and-Project Quasicrystals: Patch Frequency and Moduli Spaces

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Outline

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- Λ is uniformly discrete
  $\exists r > 0$ such that $B_r(x) \cap \Lambda = \{x\}$ for all $x \in \Lambda$
- Λ is relatively dense
  $\exists R > 0$ such that $B_R(0) + \Lambda$ covers X.

Take Λ the pointset of vertices of the tiling with two rhombuses.

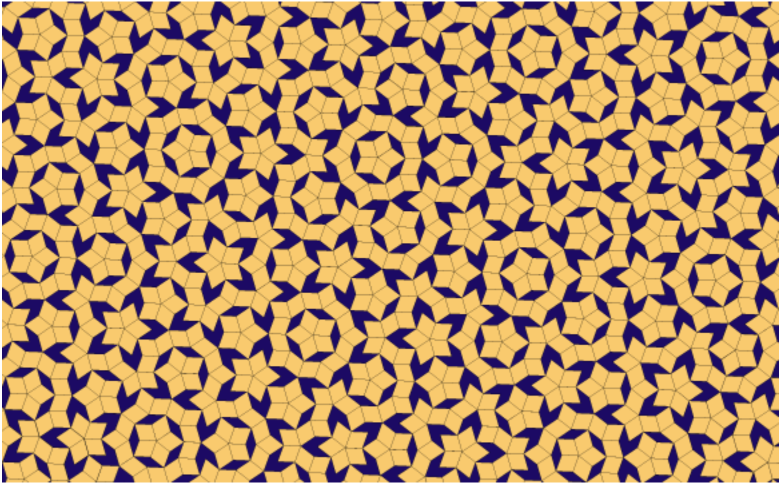

FIGURE: Penrose Rhomb '74. Source: Tilings Encyclopedia. `https://tilings.math.uni-bielefeld.de/`

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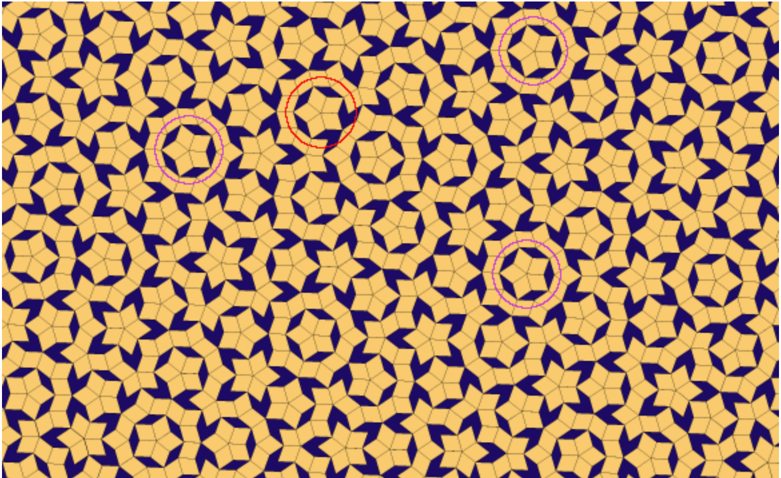

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Thm(Delone,Dolbilin,Shtogrin,Galiulin;'76)

Let $\Lambda \subset \mathbb{R}^d$ be Delone. There exists $T_0 > 0$ such that if all T_0-patches are equivalent up to isometry then there exists a lattice $\Gamma < \mathrm{Isom}(\mathbb{R}^d)$ and $x \in \Lambda$ s.t.

$$\Lambda = \Gamma x.$$

Same paper: $\mathbb{S}^n$, $\mathbb{H}^n$.

Non-examples: discrete orbits of $\mathrm{SL}_2(\mathbb{R})$

Question of Barak Weiss: For which translation surface are the set of holonomy vectors Delone?
Answer: No lattice surface is.

Theorem (Wu)

Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ a lattice and $x \in \mathbb{R}^2$ with $\Lambda = \Gamma x$ discrete.

- If Γ is arithmetic then Λ is not relatively dense
- If Γ is non-arithmetic then Λ is not uniformly discrete.

Primitive Lattice points (Baake-Moody-Pleasants)

Meyer sets

Meyer's definition of a quasicrystal

A Delone set $\Lambda \subset \mathbb{R}^d$ is called Meyer if $\Lambda - \Lambda \subset \Lambda + E$ for $E \subset \mathbb{R}^d$ finite.

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This generalizes lattices:

- If E is trivial, Λ is a lattice.
- Λ is Meyer if and only if Λ and $\Lambda - \Lambda$ are Delone. (Lagarias)

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Embedding Theorem (Meyer '72)

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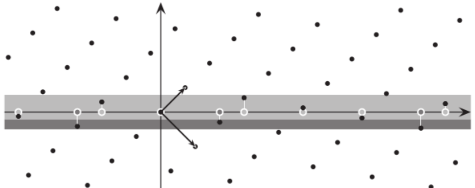

Silver mean chain. Source: APERIODIC ORDER Vol. I: A Mathematical Invitation. Baake-Grimm '13

cut-and-project sets

Definition: cut-and-project scheme

The data $(\mathcal{L}, \mathbb{R}^d, \mathcal{A})$ defines a cut-and-project scheme if

$\mathcal{A}$ is a locally compact Abelian group,

$\mathcal{L}$ is a lattice in the group $\mathbb{R}^d \times \mathcal{A}$

$\pi = \pi_{\mathbb{R}^d}, \pi_{\text{int}} = \pi_{\mathcal{A}}$ natural projections satisfy

(I) $\pi|_{\mathcal{L}}$ is injective

(D) $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathcal{A}$.

$$\begin{array}{ccccccc}
\mathbb{R}^d & \xleftarrow{\pi} & \mathbb{R}^d \times \mathcal{A} & \xrightarrow{\pi_{\text{int}}} & \mathcal{A} & \overset{\textit{window}}{\supset} & \mathcal{W} \\
\cup & & \bigvee \ \text{lattice} & & \cup \ \text{dense} & & \\
\pi(\mathcal{L}) & \overset{\text{inj}}{\hookleftarrow} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) & &
\end{array}$$

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Definition: Window and cut-and-project sets

Window $\mathcal{W} \subset \mathcal{A}$ bounded and define the cut-and-project set

$$\Lambda(\mathcal{W}, \mathcal{L}) = \pi\left(\mathcal{L} \cap \left(\mathbb{R}^d \times \mathcal{W}\right)\right)$$

If $\mathcal{W}$ has non-empty interior, call it cut-and-project quasicrystal

Minkowski Space

Meyer '72: Λ Meyer and $\theta\Lambda \subset \Lambda$ for real algebraic θ then θ is Pisot or Salem
de Bruijn '81, Pleasants '01: Penrose tilings are cut-and-project quasicrystals

S-adics

- $\mathbb{K}$ is a number field of degree N. $\mathfrak{o}$ ring of integers.
- field embeddings $\sigma_i : \mathbb{K} \to \overline{\mathbb{K}}$ with completions $\mathbb{K}_\nu$, $\nu \in S$
- $\mathbb{K}_S = \prod_{\nu \in S} \mathbb{K}_\nu$
- $\mathcal{L} = \{(\sigma_1(v), \ldots, \sigma_N(v)) : v \in \mathfrak{o}^k\} < \mathbb{K}_S^k$ lattice

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$\mathbb{R}^d \hookrightarrow \mathbb{K}_{\nu_1}^k$ for some ν_1, $\mathbb{K}_S^k = \mathbb{R}^d \oplus \mathbb{R}^m = \mathbb{R}^n$, $\mathcal{W} \subset \mathbb{R}^m$

$$\mathbb{R}^d \xleftarrow{\pi} \mathbb{K}_S^k = \mathbb{R}^n \xrightarrow{\pi_{\text{int}}} \mathbb{R}^m$$

$$\Lambda = \pi(\mathcal{L} \cap \mathbb{R}^d \times \mathcal{W})$$

Patch Counting

Sometimes assume (see Moody, Pleasants)

1. Λ is *regular* if the boundary of $\mathcal{W}$ has zero measure
2. Λ is *generic* if $\pi_{\text{int}}(\mathcal{L})$ doesn't intersect the boundary of $\mathcal{W}$.

For any T-patch $\mathcal{P} = \mathcal{P}(y, T) = \Lambda \cap B_T(y)$ define

$$\text{freq}_\Lambda(\mathcal{P}) = \lim_{t \to \infty} \frac{\#\{x \in \Lambda \cap B_t(0) : \mathcal{P}(x, T) \sim_{\mathbb{R}^d} \mathcal{P}\}}{t^d}.$$

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where $\sim_{\mathbb{R}^d}$ means equivalence up to translation.

Theorem on Patch Frequency Asymptotics

Fix $d + m = n$. There exists $\kappa > 0$ such that for a **random** cut-and-project quasicrystal Λ of $(\mathcal{L}, \mathbb{R}^d, \mathbb{R}^m)$ we have

$$\#\{x \in \Lambda \cap B_t(0) : \mathcal{P}(x, T) \sim_{\mathbb{R}^d} \mathcal{P}\} = \mathrm{freq}_\Lambda(\mathcal{P})t^d + \mathcal{O}(t^{d-\kappa}).$$

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Patches $\mathcal{P}$ of $\Lambda(\mathcal{L}, \mathcal{W})$ correspond to points in $\Lambda(\mathcal{L}, \mathcal{W}_\mathcal{P})$ for some $\mathcal{W}_\mathcal{P} \subset \mathcal{W}$. Reduces to point counting $\mathcal{L} \cap B_t(0) \times \mathcal{W}_\mathcal{P}$.

Spaces of quasicrystals

Construction of Marklof and Strömbergsson

- $\Lambda = \Lambda(\mathcal{L}, \mathcal{W})$ cut-and-project quasicrystal of $(\mathcal{L}, \mathbb{R}^d, \mathbb{R}^m)$
- Suppose $\mathcal{L} = g\mathbb{Z}^n \in X_n$ for $g \in \mathrm{SL}_n(\mathbb{R})$ and consider $\mathrm{SL}_d(\mathbb{R}) < \mathrm{SL}_n(\mathbb{R})$ top-left block.
- $\overline{\mathrm{SL}_d(\mathbb{R})\mathcal{L}} = L'\mathcal{L}$ for some $L' < \mathrm{SL}_n(\mathbb{R})$ (Ratner)

The space of quasicrystals associated to $\mathcal{L}$

$$\mathfrak{Q} = \{\Lambda(y, \mathcal{W}) : y \in L'\mathcal{L}\}$$

with probability measure $\mu_{\mathfrak{Q}}$ (push forward of $m_{L'\mathcal{L}}(y)$ under $y \mapsto \Lambda(y, \mathcal{W})$).

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MS actually consider $\mathrm{ASL}_d(\mathbb{R})$-orbit closure.
For $m_{L'\mathcal{L}}$-a.e. y defines a cut-and-project scheme i.e. it holds

(I) $\pi|_{\mathcal{L}}$ is injective

(D) $\pi_{\mathrm{int}}(\mathcal{L})$ is dense in $\mathbb{R}^m$.

What are the spaces of quasicrystals?

Identify X_n with $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$.

Classification Theorem I

Write $Y = \overline{\mathrm{SL}_d(\mathbb{R}) g\, \mathrm{SL}_n(\mathbb{Z})} = L'\mathcal{L} = gL\, \mathrm{SL}_n(\mathbb{Z})$. Then L is almost $\mathbb{Q}$-simple.

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Classification Theorem II

By the classification of semi-simple groups it follows that $\widetilde{L} \simeq \mathbb{G}(\mathbb{K}_S)$ where

- $\mathbb{G} = \mathrm{SL}_k$ for $k \geq d$ or Sp_{2k} (if $d = 2$)
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Generalizes [MS '13] $Y = X_n$ for the case $d > m$.
Unpublished work of O. Sargent.
Private communication with M. Einsiedler.

Lemma about rational invariant subspaces

(I) $\pi|_{\mathcal{L}}$ is injective

(D) $\pi_{\text{int}}(\mathcal{L})$ is dense in $\mathbb{R}^m$.

(I) and (D) implies (L) Lemma

A vector space $V < \mathbb{R}^n$ is called $\mathcal{L}$-**rational** if $V \cap \mathcal{L}$ is a lattice in V.

The following implications hold.

a. **(D)** $\Rightarrow$ $\mathbb{R}^d$ is not contained in a proper $\mathcal{L}$-rational subspace.

b. **(I)** $\Rightarrow$ $\mathbb{R}^m$ contains no non-trivial $\mathcal{L}$-rational subspace.

Define

(L) There exists no proper $\mathcal{L}$-rational subspace of $\mathbb{R}^n$ that is L'-invariant w.r.t. matrix multiplication.

c. **(I)** and **(D)** $\Rightarrow$ **(L)**.

(L) implies almost $\mathbb{Q}$-simple

Theorem

Write $Y = \overline{\mathrm{SL}_d(\mathbb{R}) g\, \mathrm{SL}_n(\mathbb{Z})} = L'\mathcal{L} = gL\,\mathrm{SL}_n(\mathbb{Z})$.
Then L is almost $\mathbb{Q}$-simple algebraic group. $\tilde{L} \simeq \mathbb{G}(\mathbb{K}_S)$ where

- *$\mathbb{G} = \mathrm{SL}_k$ for $k \geq d$ or Sp_k (if $d = 2$)*
- *$\mathbb{K}$ real number field of degree N such that $kN = n$*

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Theorem

Write $Y = \overline{\mathrm{SL}_d(\mathbb{R})g\,\mathrm{SL}_n(\mathbb{Z})} = L'\mathcal{L} = gL\,\mathrm{SL}_n(\mathbb{Z})$*.*
Then L *is almost* $\mathbb{Q}$*-simple algebraic group.* $\tilde{L} \simeq \mathbb{G}(\mathbb{K}_S)$ *where*

- $\mathbb{G} = \mathrm{SL}_k$ *for* $k \geq d$ *or* Sp_k *(if* $d = 2$*)*
- $\mathbb{K}$ *real number field of degree* N *such that* $kN = n$

Proof.

- Shah '91: L minimal $\mathbb{Q}$-group generated by unipotents containing $H' = g^{-1}\,\mathrm{SL}_d(\mathbb{R})g$
- L is semi-simple: Let U denote the unipotent radical of L. V^U is a rational subspace. Cannot be by Lemma.
- H' must be contained in an almost direct factor (over $\mathbb{R}$). It follows that L is almost $\mathbb{Q}$-simple
- Use classification (see Tits '66, Morris-Witte '14)

□

Theorem (in Tits '66)

If L is almost $\mathbb{Q}$-simple and simply connected, there exists a field $\mathbb{K}$ and an absolutely almost simple simply connected group G defined over $\mathbb{K}$ such that $L \simeq_{\mathbb{Q}} Res_{\mathbb{K}/\mathbb{Q}}(G)$.

Theorem (Morris-Witte in Appendix of Solomon-Weiss '14)

If L as above contains a conjugate of the top left $\mathrm{SL}_d(\mathbb{R})$ then G is either of type A_k or C_k.

Siegel formulas

Siegel's formula

Theorem (Siegel's mean value theorem '45)

Let $f : \mathbb{R}^d \to \mathbb{R}$ is Riemann integrable, bounded, compactly supported. Then

$$\int_{X_d} \sum_{x \in \Lambda - \{0\}} f(x) dm_{X_d}(\Lambda) = \int_{\mathbb{R}^d} f(y) dy.$$

Also r-tuples (of rank r), number fields and symplectic group (no proofs).

Siegel measure

$\mathcal{M}_d = \left\{ \nu \in \text{Radon}(\mathbb{R}^d) : \sup_t \frac{\nu(B_t(0))}{t^d} < \infty \right\}$
For Λ Delone, $\nu_\Lambda = \sum_{x \in \Lambda - \{0\}} \delta_x \in \mathcal{M}_d$. Action by $\text{SL}_d(\mathbb{R})$.

Veech's definition of a Siegel measure

An $\text{SL}_d(\mathbb{R})$-invariant ergodic probability measure μ on $\mathcal{M}_d$ is called Siegel measure.

$\mu_{\mathfrak{Q}}$ is a Siegel measure supported on space of quasicrystals (seen as point set distribution in $\mathcal{M}_d$).

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Theorem (Veech '98, MS '13 for quasicrystals)

For any Siegel measure μ there exists c_μ such that for all $f \in L^1(\mathbb{R}^d)$

$$\int \nu(f) d\mu(\nu) = c_\mu \int_{\mathbb{R}^d} f(y) dy.$$

Ergodic theoretic proof.

Rogers' formula

Write $y = (y_1, \ldots, y_s) \in (\mathbb{R}^d)^s$. $\widehat{h}(\Lambda) = \nu_\Lambda^{\otimes r}$. $s \leq r < d$.

Rogers' formula on r-Moments '55

Let $h : (\mathbb{R}^d)^r \to \mathbb{R}$ non-negative Borel measurable

$$\int_{X_n} \widehat{h}(\Lambda)\, dm_{X_d}(\Lambda) = \int_{X_n} \sum_{x_1,\ldots,x_r \in \Lambda \setminus \{0\}} h(x_1, \ldots, x_r)\, dm_{X_d}(\Lambda)$$

$$= \sum_{(\nu;\mu)} \sum_{q=1}^{\infty} \sum_{D} c_{q,D} \int_{(\mathbb{R}^d)^s} h\left(\frac{1}{q} yD\right) dy.$$

- *admissible* $D \in \mathrm{Mat}_{sr}(\mathbb{Z})$ (rank s) w.r.t. partition $(\nu; \mu)$
- $c_{q,D}^{1/d} = \prod \gcd(\varepsilon_i, q)$; ε_i elementary divisors of D

Schmidt '57,'58,'59. Macbeath-Rogers '57.
Better understanding of formula: Strömbergsson-Södergren '19
RHS is sum of measures on rank s subspaces of $(\mathbb{R}^d)^r$

Integrability and Riesz-Representation Thm

- $X = (\mathbb{K}_S^d)^r$ with $L = \mathrm{SL}_d(\mathbb{K}_S)$-action. $L_x = \mathrm{Stab}(x)$ for $x \in X$.
- any L-ergodic measure on X is supported on $L_x.x \simeq L/L_x$
- $\mu \in \mathrm{Prob}(\mathrm{Radon}(X))$ L-ergodic (Siegel measure)
- $f \in B_c(\mathbb{K}_S^d)$, $h \in B_c(\mathbb{K}_S^d)^{\otimes r}$

Disintegration of Siegel measures

Let $h = f^{\otimes r}$. If $\nu \mapsto \nu(f) \in L^r(\mu)$ then $\widehat{h}(\nu) = \nu^{\otimes r}(h) \in L^1(\mu)$. Hence $\tau : h \mapsto \int \hat{h} d\mu$ defines a measure on X and

$$\mu(\hat{h}) = \tau(h) = \int_{x \in X/L} \tau_x(h) d\tilde{\mu}(x)$$

where τ_x is a L-ergodic measure on L/L_x.

$r = 1$: Veech '98 for $\mathrm{SL}_n(\mathbb{R})$, Venkatesh '13 for $\mathrm{SL}_2(\mathbb{K}_S)$.
$r = 2$: Athreya-Cheung-Masur '19 for $\mathrm{SL}_2(\mathbb{R})$.

Integrability Lemma

Integrability Lemma

Let $f : \mathbb{K}_S^d \to \mathbb{R}$ bounded compactly supported.
If $L = \mathrm{SL}_d(\mathbb{K}_S)$ then $\sum_{v \in \ell \mathfrak{o}^d} f(v)$ is in $L^r(L/\Gamma)$ for $r < d$.
If $L = \mathrm{Sp}_{2d}(\mathbb{K}_S)$ then $\sum_{v \in \ell \mathfrak{o}^{2d}} f(v)$ is in $L^r(L/\Gamma)$ for $r < d + 1$.

Proof via reduction theory:
$\mathrm{SL}_d(\mathbb{R})$ - Lemma 3.10 in Eskin-Margulis-Mozes '98.
$\mathrm{SL}_2(\mathbb{K}_S)$ - Venkatesh '13.

Weil's formula

Weil's Modification of Fubini Theorem ('46)

- $G' < G$ both lcsc unimodular
- $\Gamma < G$ lattice
- $\Gamma' = \Gamma \cap G'$ lattice
- $f \in L^1(G/G')$ and $\hat{f}(g\Gamma) = \int_{\Gamma/\Gamma'} f(g\xi)d\xi$

Then $\hat{f} \in L^1(G/\Gamma)$ and

$$\int_{G/\Gamma} \hat{f}\, dm_{G/\Gamma} = c \int_{G/G'} f dm_{G/G'}.$$

Weil's book '61. $c = \mathrm{vol}(G'/\Lambda')$

Decomposition of Orbits

Get more information from Weil's theorem

$$\Lambda = \sum \Lambda_x, \quad \Lambda_x = g\,\mathrm{SL}_d(\mathfrak{o})x, \quad x \in (\mathfrak{o}^d)^r$$

$$\mu(\hat{h}) = \sum_{\mathrm{SL}_d(\mathfrak{o}).x} \mu\left(\sum_{x' \in \mathrm{SL}_d(\mathfrak{o}).x} h(x')\right) =_{\mathrm{def}} \sum_{x \in X_{\mathfrak{o}}/\mathrm{SL}_d(\mathfrak{o})} \mu({}^x\hat{h})$$

By Weil's formula

$$= \mu(\hat{h}) = \sum \mu({}^x\hat{h}) = \sum c_x \int_{L/L_x} h dm_{L/L_x} = \sum \tau_x(h)$$

$L.x$ open dense in some subspace $V_x = (\mathbb{K}_S^d)^r$. Identify τ_x with Haar of V_x.

Generalized Rogers Formula

$$\mu(\widehat{(1_{B_t})^{\otimes r}}) = \sum_{\text{orbits } L.x} c_x t^{\kappa_x}$$

is a polynomial in t and leading coefficient $= c_\mu$.

Spherical symmetrization

Upgrade from balls to general f by spherical symmetrization.

Rogers' Spherical symmetrization '55, '56 (special case)

For any f, define its spherical symmetrization by $f^* = 1_{B_t}$ where t is chosen such that $\int_{\mathbb{R}^n} f(y)dy = m_{\mathbb{R}^n}(B_t)$.

$$\int_{\mathbb{R}^n} f(ay)f(by)dy \leq \int_{\mathbb{R}^n} f^*(ay)f^*(by)dy$$

Groups of Rogers type

Bounds on second moments

Let $G < \mathrm{SL}_n(\mathbb{R})$ be algebraic with diagonal action on $\mathbb{R}^n \times \mathbb{R}^n$. Let $Y = g_0 G\, \mathrm{SL}_n(\mathbb{Z})$ be a closed orbit with probability measure m_Y. Suppose

$$\hat{f}(y) = \sum_{v \in y\mathbb{Z}^n - 0} f(v) \in L^2(m_Y) \text{ for any } f \in C_c(\mathbb{R}^n).$$

and spherical symmetrization applies (*). There exists $c' < \infty$ such that for any $f \in L^1(\mathbb{R}^n)$ with $V = c_\mu m_{\mathbb{R}^n}(f)$ we have

$$\int \left(\hat{f} - V\right)^2 dm_Y = c' V.$$

Schmidt '60 for $\mathrm{SL}_n(\mathbb{R})$. Kelmer-Yu '19 for $\mathrm{Sp}_{2n}(\mathbb{R})$
Randol'70/Kelmer-Mohammadi'12/Athreya-Konstantoulas'16
(*) We don't know this in general for the number field setting

Patch frequency

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Thank you.