A SERIES OF ARTICLES ON ERGODIC THEORY

A summer school on ergodic theory with about eighty participants was held in the second half of September 1965 at the Khumsan State University and Institute of Mathematics of the Academy of Sciences UzSSR, near Tashkent. The articles of this series arise from lectures given at that school. An exception is V.A. Rokhlin's article, which was written earlier.

LECTURES ON THE ENTROPY THEORY OF MEASURE-PRESERVING TRANSFORMATIONS

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Contents

Introduction

This article reproduces, with some additions, the second part of a course of lectures at Leningrad University in the academic year 1962/63. Its topic is a new development in ergodic theory connected with the concept of entropy. The first part of the course was concerned with more classical problems fully covered by the literature. Lecture notes by R.M. Belinskii, S.M. Belinskii, and S.A. Yuzvinskii serve as the basic text.

For the convenience of readers the article begins with a short account of the necessary concepts of measure theory, classical ergodic theory, and the spectral theory of operators. A detailed account can be found in articles by the author [20], [21], Halmos's book [37], and Plesner's book [19]. Some statements, not explicitly contained in these works, are proved. These preliminaries take up the first three sections.

LITERATURE. The theory expounded in this article is built up in papers of Kolmogorov [13], [14], Sinai [30], Pinsker [17], Sinai and the author [29], and the author [23], [25], [27]. Roughly, it can be described as the general entropy theory of dynamical systems with discrete time. Some important results, also concerned with this subject, are not contained in this article; for example, Sinai's theorem on weak isomorphism (see [32], [34]) is not proved, Abramov's theorem on the entropy of the derived automorphism (see [1]) and the theorem by Abramov and the author on the entropy of a skew product are not even mentioned (see [1] and [5]). The list of results not covered by this article would be much longer if we were to consider all the entropy theory of measure-preserving transformations with invariant measure. It contains no general entropy theory for flows ([14], [2], [31], [11]), nor the entropy theory of transformations and flows that occur in neighbouring domains of mathematics: the classical theory of dynamical systems (see [33], [35], [36], [7], [10], [15]), probability theory (see [17], [18]), number theory (see [25], [50]), and topological algebra (see [30], [3], [28], [8], [38], [40]). The reader who wishes to extend his knowledge, must turn to the works listed (see also Yuzvinskii's Appendix to this article and other articles in the series).

In conclusion, I must point out that there are already in the literature monographs dealing, partly or entirely, with the entropy theory of measure-preserving transformations: the survey articles [24], [45], [33], [47] and the text-books [43], [46], [42], [51].

§1. Preliminaries from measure theory

1.1. It is assumed that the reader is familiar with general measure theory. The measures that we shall encounter, are complete (that is, subsets of sets of zero measure are measurable and have zero measure) and normalized (that is, the measure of the whole space is 1)

A map from one measure space to another is said to be a *homomorphism* if the inverse image of a measurable set is measurable and has the same measure. A homomorphism is said to be an *isomorphism* if it is one-to-one and the inverse map is also a homomorphism. If the spaces coincide then an isomorphism is said to be an *automorphism* and an homomorphism an *endomorphism*.

Two measure spaces are said to be *isomorphic* if an isomorphism can be constructed between them. An endomorphism T in a space M is said to be *isomorphic* to an endomorphism T_1 in a space M_1 if there exists an isomorphism S from M to M_1 such that $T_1 = STS^{-1}$.

The most important principle of measure theory is that of neglecting sets of measure zero. In accordance with this principle measurable sets and endomorphisms are considered to within sets of zero measure or as we say '*modulo* 0' (mod 0). For example, spaces M and M_1 or their endomorphisms T and T_1 could be non-isomorphic, but could be made isomorphic by removing from M or M_1 some set with measure 0, or an endomorphism T of a space M that is not an automorphism might be made to be one by removing a set of measure 0 from M. The addition 'mod 0' is often implied without being explicitly stated.

1.2. A countable system $\{B_\alpha;\ \alpha \in A\}$ of measurable sets is said to be a *basis* of M if:

a) for any measurable set X, there exists a set Y in the system of Borel sets generated by $\{B_\alpha;\ \alpha \in A\}$ such that $Y \supset X$ and $\mu(Y - X) = 0$;

b) for any two points $x \in M$, $y \in M$ there exist an $\alpha \in A$ such that either $x \in B_\alpha$, $y \notin B_\alpha$, or $x \notin B_\alpha$, $y \in B_\alpha$.

A space with a complete normalized measure and a basis is said to be *separable*. A separable space M is said to be *complete* with respect to its basis $\{B_\alpha;\ \alpha \in A\}$ if all intersections $\bigcap_{\alpha \in A} E_\alpha$, where E_α is either B_α or $M - B_\alpha$, are non-empty. A separable space M is said to be *complete mod* 0 with respect to its basis $\{B_\alpha;\ \alpha \in A\}$ if it is a subspace of measure 1 of a space M' that is complete with respect to its basis $\{B'_\alpha;\ \alpha \in A\}$, such that $B'_\alpha \cap M = B_\alpha$. If a separable space is complete mod 0 with respect to one basis, it is complete mod 0 with respect to all bases. Such spaces are called *Lebesgue spaces* and their measures, *Lebesgue measures*.

A Lebesgue space contains at most a countable set of points of positive measure. If this set is empty, the measure is said to be *continuous*; if it exhausts (mod 0) the whole space, the measure is said to be *discrete*. A Lebesgue space with a continuous measure is isomorphic mod 0 to the unit interval with the usual Lebesgue measure. The product of a finite or countable collection of Lebesgue spaces is a Lebesgue space.

A homomorphism of a Lebesgue space onto a Lebesgue space takes every measurable set that is an inverse image into a measurable set. In particular, a one-to-one homomorphism is an isomorphism. However, a measurable set that is not inverse images can be taken by the homomorphism into a non-measurable set.

If M is a Lebesgue space and C is a measurable set in M with $\mu(C) > 0$, then the formula $\mu_c(X) = \dfrac{\mu(X)}{\mu(C)}$, where X is a set contained in C and measurable in M, turns C into a Lebesgue space with measure μ_c. This space is said to be a *subspace* of M.

Henceforth it is assumed that all subspaces are Lebesgue.

1.3. Any collection of non-empty disjoint sets that cover M is said to be a *partition* of M. Subsets of M that are sums of elements of a partition ζ is called *ζ-sets*.

A countable system $\{B_\alpha;\ \alpha \in A\}$ of measurable ζ-sets is said to be a *basis* of ζ if, for any two elements C and C' of ζ, there exists an $\alpha \in A$ such that either $C \subset B_\alpha$, $C' \not\subset B_\alpha$ or $C \not\subset B_\alpha$, $C' \subset B_\alpha$. A partition with a basis is said to be *measurable*.

We write $\zeta \leqslant \zeta'$ if ζ' is a subpartition of ζ. $\zeta \leqslant \zeta'$ and $\zeta = \zeta'$ are both considered up to mod 0.

For any system of measurable partitions $\{\zeta_\alpha\}$ there exists a *product* $\bigvee_\alpha \zeta_\alpha$, defined as the measurable partition ζ satisfying the two conditions: $\zeta_\alpha \leqslant \zeta$ for all α; if $\zeta_\alpha \leqslant \zeta'$ for all α, then $\zeta \leqslant \zeta'$. The product $\bigvee_{i=1}^{n} \zeta_i$ is also denoted $\zeta_1\zeta_2 \ldots \zeta_n$.

For any system of measurable partitions $\{\zeta_\alpha\}$, there exists an *intersection* $\bigwedge_\alpha \zeta_\alpha$, defined as the measurable partition ζ satisfying the two contions: $\zeta_\alpha \geqslant \zeta$ for all α; if $\zeta_\alpha \geqslant \zeta'$ for all α, then $\zeta \geqslant \zeta'$.

The symbol $\zeta_n \nearrow \zeta$ indicates that $\zeta_1 \leqslant \zeta_2 \leqslant \ldots$ and $\bigvee_{n=1}^{\infty} \zeta_n = \zeta$. The symbol $\zeta_n \searrow \zeta$ indicates that $\zeta_1 \geqslant \zeta_2 \geqslant \ldots$ and $\bigwedge_{n=1}^{\infty} \zeta_n = \zeta$.

A partition of M into distinct points is denoted by ε. The trivial partition, having the single element M, is denoted by ν.

If $B_1, B_2, \ldots$ is a basis for the partition ζ and β_n is the partition of M into the sets B_n and $M - B_n$, then the partitions $\zeta_n = \beta_1\beta_2 \ldots \beta_n$ form an increasing sequence and $\bigvee \zeta_n = \zeta$. Thus, for any measurable partition ζ there exists a sequence of finite partitions ζ_n such that $\zeta_n \nearrow \zeta$. Measurable partitions ξ and η are said to be *independent* if $\mu(A \cap B) = \mu A \cdot \mu B$ for any measurable ξ-set A and any measurable η-set B.

The set B is called *independent* of the measurable partition ξ if the above equation holds for any measurable ξ-set A.

A function f, defined on M, is said to be *independent* of ξ if all its Lebesgue sets are independent of ξ.

1.4. From the collection of all measurable sets we obtain classes of sets, the elements of each class being equal mod 0, and we denote the set of classes by $\mathfrak{M}$. The operations of countable union, countable intersection and subtraction of sets goes over to the same operations on classes, making $\mathfrak{M}$ an *algebra*. Any part of $\mathfrak{M}$ that is closed with respect to these operations is said to be a *subalgebra* of $\mathfrak{M}$.

It is clear that the *intersection* $\bigcap_\alpha \mathfrak{M}_\alpha$ of any system of subalgebras $\mathfrak{M}_\alpha$ of $\mathfrak{M}$ is a subalgebra of $\mathfrak{M}$. The *sum* $\bigvee_\alpha \mathfrak{M}_\alpha$ of subalgebras $\mathfrak{M}_\alpha$ is defined to be the intersection of all subalgebras that (each) contain all the $\mathfrak{M}_\alpha$. If $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \ldots$, and $\bigvee \mathfrak{M}_n = \mathfrak{M}'$, then we write $\mathfrak{M}_n \nearrow \mathfrak{M}'$. If $\mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \ldots$, and $\bigcap \mathfrak{M}_n = \mathfrak{M}'$, then we write $\mathfrak{M}_n \searrow \mathfrak{M}'$.

Among the subalgebras of $\mathfrak{M}$ there is a largest – $\mathfrak{M}$ itself, and a smallest – the trivial algebra $\mathfrak{N}$ consisting of the class of sets of measure 0 and the class of sets of measure 1.

For any measurable partition ξ, we denote by $\mathfrak{M}(\xi)$ the subalgebra of $\mathfrak{M}$

consisting of classes of measurable ξ-sets. If $\mathfrak{M}(\xi)=\mathfrak{M}(\xi')$, then $\xi=\xi'$, and for any subalgebra of $\mathfrak{M}$ there exists a measurable partition ξ such that $\mathfrak{M}(\xi)$ is the subalgebra. Thus, the subalgebras of $\mathfrak{M}$ are in one-to-one correspondence with the classes of (mod 0)-equal measurable partitions. Here $\mathfrak{M}(\xi)\subset\mathfrak{M}(\xi')$ if and only if $\xi\leqslant\xi'$,

$$\mathfrak{M}(\bigvee_\alpha \xi_\alpha)=\bigvee_\alpha \mathfrak{M}(\xi_\alpha), \qquad \mathfrak{M}(\bigwedge_\alpha \xi_\alpha)=\bigcap_\alpha \mathfrak{M}(\xi_\alpha).$$

In particular, $\mathfrak{M}(\xi_n)\nearrow\mathfrak{M}(\xi)$ is equivalent to $\xi_n\nearrow\xi$ and $\mathfrak{M}(\xi_n)\searrow\mathfrak{M}(\xi)$ to $\xi_n\searrow\xi$. We note that $\mathfrak{M}(\varepsilon)=\mathfrak{M}$, $\mathfrak{M}(\nu)=\mathfrak{N}$.

The *distance* $\rho(A,B)$ between measurable sets A and B is defined by the formula $\rho(A,B)=\mu((A\bigcup B)-(A\bigcap B))$. The function ρ makes $\mathfrak{M}$ a complete separable metric space. The operations of union, intersection and subtraction are continuous with respect to this metric.

1.5. The *factor-space* of M with respect to ζ is the measure space whose points are the elements of ζ and the measure μ_ζ defined as follows: let p be the map taking each point $x\in M$ to the element of ζ in which it is contained; a set Z is considered to be measurable if $p^{-1}(Z)$ is measurable in M, and we define $\mu_\zeta(Z)=\mu(p^{-1}(Z))$. We denote this factor-space by M/ζ. It is clear that p is a homomorphism of M onto M/ζ. This natural homomorphism is said to be a *projection*. The factor-space of a Lebesgue space with respect to a measurable partition is a Lebesgue space.

1.6. As usual, we denote by $L_2(M)$ the unitary space of square integrable functions on M; by (f,g) the scalar product of $f,\ g\in L_2(M)$; by $\|f\|$ the norm of f.

For any measurable partition ζ we denote by $L_2(M,\ \zeta)$ the subspace of $L_2(M)$ consisting of the functions that are constant on the elements of ζ. $L_2(M,\ \zeta)$ contains the characteristic functions of the sets of $\mathfrak{M}(\zeta)$ and is generated by these functions. It follows that $L_2(M,\ \zeta)=L_2(M,\ \zeta')$ if and only if $\zeta=\zeta'$. It is also clear that $L_2(M,\ \zeta)\subset L_2(M,\ \zeta')$ if and only if $\zeta\leqslant\zeta'$, and that $\zeta_n\nearrow\zeta$ is equivalent to

$$L_2(M,\ \zeta_n)\nearrow L_2(M,\ \zeta), \tag{1}$$

and $\zeta_n\searrow\zeta$ is equivalent to

$$L_2(M,\ \zeta_n)\searrow L_2(M,\ \zeta). \tag{2}$$

Formula (1) means, of course, that

$$L_2(M,\ \zeta_1)\subset L_2(M,\ \zeta_2)\subset\ldots,\qquad \bigvee L_2(M,\ \zeta_n)=L_2(M,\ \zeta),$$

and formula (2) that

$$L_2(M,\ \zeta_1)\supset L_2(M,\ \zeta_2)\supset\ldots,\qquad \bigcap L_2(M,\ \zeta_n)=L_2(M,\ \zeta).$$

It is also clear that $L_2(M,\ \varepsilon)=L_2(M)$, $L_2(M,\ \nu)=C(M)$, where $C(M)$ is the one-dimensional subspace of constants.

For any measurable partition ζ, $L_2(M,\ \zeta)$ is canonically isomorphic to $L_2(M/\zeta)$: the function $f\in L_2(M/\zeta)$ corresponding to the function $g\in L_2\ (M,\zeta)$ defined by $g(x)=f(px)$, $x\in M$, where p is the projection.

1.7. A *canonical system of measures* or a system of *conditional* measures, belonging to a partition ζ, is a system of measures $\{\mu_C\}$, $C \in \zeta$, satisfying the two conditions:

a) μ_C is a Lebesgue measure on C, $C \in \zeta$;

b) for any measurable set $X \subset M$, the set $X \cap C$ is measurable in C for almost all points $C \in M/\zeta$. The function $\mu_C(X \cap C)$ is measurable on M/ζ for

$$\mu(X) = \int_{M/\zeta} \mu_C(X \cap C)\, d\mu_\zeta.$$

Every measurable partition has a canonical system of measures, and any two systems $\{\mu_C\}$ and $\{\mu'_C\}$ belonging to the same partition ζ are identical mod 0 (that is, $\mu'_C = \mu_C$ for almost all $C \in M/\zeta$).

If $\mu(C) > 0$, then μ_C coincides with the measure in C defined in 1.2.

If ζ is a measurable partition and C is a subspace of M, then we denote by ζ_c the partition of C induced by ζ. Generally, ζ_C denotes the partition induced by ζ in the element C of another measurable partition ζ (regarded as a Lebesgue space with measure μ_c). ζ_C is measurable.

Let ξ and ζ be measurable partitions such that $\xi \geqslant \zeta$ and let A be an element of ξ and C an element of ζ containing A. As an element of the partition ξ of M, A is a Lebesgue space with measure μ_A. On the other hand, as an element of the partition ξ_C of C with measure μ_C, A is a Lebesgue space with measure $(\mu_C)_A$. The uniqueness of canonical systems of measures implies that $(\mu_C)_A = \mu_A$ for almost all $A \in M/\zeta$. This property is called the *transitivity* of a canonical system of measures.

Measurable partitions ξ and η are said to be *independent relative* to ζ if, for almost all $C \in M/\zeta$, the partitions ξ_C and η_C are independent.

1.8. From the definition of a canonical system of measures it follows that if a (complex) function f is integrable on M, then for almost all $C \in M/\zeta$ the section f_C defined by the formula

$$f_C(x) = f(x) \text{ if } x \in C,$$

is integrable on C and

$$\int_M f(x)\, d\mu = \int_{M/\zeta} d\mu_\zeta \int_C f_C(x)\, d\mu_C.$$

If $f \in L_2(M)$, then the inner integral is in $L_2(M/\zeta)$ (Schwarz's inequality), and the function

$$g(x) = \int_{p(x)} f_{p(x)}(y)\, d\mu_{p(x)},$$

corresponding to it in the canonical isomorphism between $L_2(M/\zeta)$ and $L_2(M, \zeta)$ can be considered as the result of averaging f on the elements of ζ. The operator $E_\zeta : L_2(M) \to L_2(M, \zeta)$ defined by $E_\zeta f(x) = g(x)$ is called the *averaging operator* on ζ.

We show that E_ζ is an orthogonal projection operator onto $L_2(M, \zeta)$. Since for any function $f \in L_2(M)$

$$f = E_\zeta f + (f - E_\zeta f), \qquad E_\zeta f \in L_2(M, \zeta),$$

it is sufficient to prove that, for any $f \in L_2(M)$, the function $f - E_\zeta f$ is orthogonal to $L_2(M, \zeta)$, that is, $(f - E_\zeta f, g) = 0$ if $f \in L_2(M)$, $g \in L_2(M, \zeta)$. Let f_1 and g_1 be the functions in $L_2(M/\zeta)$ related to $E_\zeta f$ and $\overline{g}$ by the canonical isomorphism between $L_2(M, \zeta)$ and $L_2(M/\zeta)$. We have

$$(f, g) = \int_{M/\zeta} d\mu_\zeta \int_C f(x)\overline{g(x)}\, d\mu_C = \int_{M/\zeta} g_1(C)\, d\mu_\zeta \int_C f(x)\, d\mu_C =$$

$$= \int_{M/\zeta} g_1(C) f_1(C)\, d\mu_\zeta = \int_M \overline{g(x)} E_\zeta f(x)\, d\mu = (E_\zeta f, g).$$

1.9. Let $\zeta_1, \zeta_2, \ldots$ be a sequence of measurable partitions of M. According to **1.6** if $\zeta_n \nearrow \zeta$, then $L_2(M, \zeta_n) \nearrow L_2(M, \zeta)$ and if $\zeta_n \searrow \zeta$, then $L_2(M, \zeta_n) \searrow L_2(M, \zeta)$. Comparing this with **1.8**, we see that in both cases $E_{\zeta_n} f \to E_\zeta f$ in $L_2(M)$ for any function $f \in L_2(M)$.

We denote by p_n and p the projection operators from M onto M/ζ_n and M/ζ. If f is the characteristic function of a set $X \subset M$, then

$$E_{\zeta_n} f(x) = \mu_{p_n(x)}(X \cap p_n(x)), \qquad E_\zeta f(x) = \mu_{p(x)}(X \cap p(x)).$$

Consequently, if $\zeta_n \nearrow \zeta$ or $\zeta_n \searrow \zeta$, then, for any set $X \subset M$ there is a sequence of functions $\mu_{p_n(x)}(X \cap p_n(x))$ tending to $\mu_{p(x)}(X \cap p(x))$ in $L_2(M)$.

1.10. For any measurable partition η with discrete conditional measures μ_B (that is, with finite or countable mod 0 elements B) there exists a finite or countable measurable partition ζ such that $\eta\zeta = \varepsilon$. Moreover, ζ can be chosen such that for an indexing $C_1, C_2, \ldots$ of its elements the conditional measures of the single point intersections $C_i \cap B$ of the elements with each element B of η form a decreasing sequence:

$$\mu_B(C_1 \cap B) \geqslant \mu_B(C_2 \cap B) \geqslant \ldots$$

§2. Isometric operators

2.1. Let H be a separable unitary space. An *isometric operator* in H is an isomorphic transformation of H onto a subspace. An isometric operator U acting in H is said to be *unitary* if $UH = H$ and *semi-unitary* if UH is a proper subspace of H. The dimension of the complement in $H \ominus UH$ is called the *defect* of U. A subspace G of H is said to be *invariant* under U if $UG \subset G$, and *completely invariant* if $UG = G$. The operator $U|G$ induced in an invariant subspace G by an isometric operator U is also isometric. It is called a *part* of U and is unitary if and only if G is completely invariant. If H is the orthogonal sum of invariant subspaces H_i, then we say that U is the orthogonal sum of the $U_i = U|H_i$, and we write: $U = \bigoplus U_i$.

Operators U and U' defined in unitary spaces H and H' are said to be *isomorphic* if there exists an isomorphic transformation V from H to H' such that $U' = VUV^{-1}$.

2.2. In what follows we need Lebesgue-Stieltjes measures on the unit circle $C = \{z\colon |z| = 1\}$ in the complex z-plane. By definition, two such measures belong to the same *spectral type* if they are absolutely continuous with respect to each other. The relation of absolute continuity transferred from the measures to their types is called *subordination* and is denoted by the symbol $\leqslant$. If $\rho < \sigma$ (that is, $\rho < \sigma$ and $\rho \neq \sigma$), then we say that the type ρ is *strictly subordinate* to the type σ. This relationship of subordination defines a (partial) order in the set of types. Any finite or countable set of types has an upper and a lower bound, namely the *sum* and the *intersection*. In the set of all spectral types there is a smallest type 0, the type of the measure $\rho = 0$. The type of the usual Lebesgue measure is said to be *Lebesgue* and is denoted by λ.

If U is a *unitary* operator acting in H, then for any vector $f \in H$ there exists a unique Lebesgue - Stieltjes measure ρ_f on C such that

$$(U^n f, f) = \int_C z^n \, d\rho_f \qquad (n = 0, \pm 1, \ldots).$$

The spectral type of this measure is denoted by ρ_f.

2.3. For every vector f of a space H in which a unitary operator U acts there is a minimal completely invariant subspace $H(f)$ containing f, which we call the *cyclic subspace generated by the vector f*. If $H(f) = H$, then f is said to be a *generator* of the operator U. An operator having a generator is said to be *cyclic*.

If f and g are generators of U, then $\rho_f = \rho_g$; in other words, all generators of a cyclic operator U give the same spectral type. This is said to be the *spectral type* of the cyclic operator and is denoted by $\rho(U)$. It is maximal: spectral types corresponding to other (non-generating) vectors are strictly subordinate to $\rho(U)$.

2.4. Every unitary operator can be expanded as the orthogonal sum of cyclic parts $U_1, U_2, \ldots$ satisfying the condition

$$\rho(U_1) \geqslant \rho(U_2) \geqslant \ldots \tag{3}$$

It is convenient to assume that the sequences $U_1, U_2, \ldots$ and $\rho(U_1), \rho(U_2), \ldots$ are infinite: if they are finite, we add zeros. From the theory of unitary operators we know that a sequence $\rho(U_1), \rho(U_2), \ldots$ satisfying (3) does not depend on the choice of the expansion $U = \bigoplus U_i$. We put $\rho_i(U) = \rho(U_i)$ and call $\rho_1(U), \rho_2(U), \ldots$ the *spectral sequence* of U. It follows from what we have said that operators U and V are isomorphic if and only if $\rho_n(U) = \rho_n(V)$ $(n = 1, 2, \ldots)$.

2.5. A unitary operator U has, by definition, a *simple spectrum* if $\rho_2 = 0$ and a *multiple spectrum* if $\rho_2 \neq 0$. Clearly, an operator has a simple spectrum if and only if it is cyclic.

The *multiplicity* of a non-zero spectral type ρ is the number of elements of the spectral sequence that are subordinate to ρ. We denote by $\rho_\infty(U)$ the intersection of all non-zero elements of the spectral sequence. If $\rho_\infty(U) \neq 0$, the multiplicity of ρ_∞ is said to be the multiplicity of the spectrum of U (if $\rho_\infty(U) = 0$ the multiplicity of the spectrum is not defined). The spectrum is said to be *homogeneous* if all the non-zero elements of the spectral sequence are equal to each other. If all the elements are subordinate to λ, the spectrum is said to be *absolutely continuous*. If all the terms of a spectral sequence are singular (that is, the corresponding measures are singular with respect to λ), the spectrum is said to be *singular*. If all the terms are discrete (that is, are types of discrete measures), the spectrum is said to be *discrete* (this is equivalent to the requirement that the orthogonal sum of the eigenspaces is H). If all the elements are continuous (that is, are types of continuous measures), the spectrum is said to be *continuous*. Continuity of the spectrum is equivalent to U not having any eigenvalues.

As an example of a homogeneous continuous spectrum we can take the *Lebesgue* spectrum. We say that a unitary operator U has a Lebesgue spectrum if all the non-zero elements of its spectral sequence are equal to λ. An equivalent definition: the space H is the orthogonal sum of completely invariant subspaces and in each of them there is a complete orthogonal system $\{f_n,\ n = 0, \pm 1, \dots\}$, such that $Uf_n = f_{n+1}$.

2.6. As an example of a semi-unitary operator we can take the operator defined by the formula $Uf_n = f_{n+1}$, where $f_0, f_1, \dots$ is a complete orthogonal system in H. This is an *elementary semi-unitary operator*. An orthogonal sum of p elementary semi-unitary operators (p being an integer or ∞) is said to be a *semi-unitary operator with a homogeneous spectrum of multiplicity* p. Clearly, the defect of such an operator is p.

Turning to an arbitrary semi-unitary operator we consider the intersection $H^0 = \bigcap_{n=0}^{\infty} U^n H$. From the obvious inclusions $H \supset UH \supset U^2 H \supset \dots$ it follows that this intersection is completely invariant and that $U|H^0$ is a unitary operator. The orthogonal complement $H^1 = H \ominus H^0$ is also invariant under U and, clearly,

$$\bigcap_{n=0}^{\infty} U^n H^1 = 0, \tag{4}$$

where 0 is the zero subspace. On H^1 the operator U has a homogenous spectrum.

For let $\{h_\alpha\}$ be a complete orthogonal system in $H^1 \ominus UH^1$ and let H_α be the closed linear hull of the sequence $h_\alpha, Uh_\alpha, U^2 h_\alpha, \dots$ Clearly, the vectors $U^n h_\alpha$ are pairwise orthogonal. Consequently, the subspaces H_α are pairwise orthogonal and in each of them U is an elementary semi-unitary operator. By (4) the system $\{U^n h_\alpha\}$ is complete in H^1. Thus, the orthogonal sum of the H_α is the whole of H^1.

We note that $H^1 \ominus UH^1 = H \ominus UH$, so that the number of subspaces H_α is equal to the defect of U.

So we have shown that for any semi-unitary operator U the space H is the orthogonal sum of invariant subspaces H^0 and H^1, in the first of which U is unitary while in the second U has a homogenous spectrum whose multiplicity is equal to the defect of U.

The operators induced by U in H^0 and H^1 are said to be the *unitary part* and the *homogenous part* of U.

It follows from what we have said that two semi-unitary operators are isomorphic if and only if their defects are equal and their unitary parts are isomorphic.

§3. Measure-preserving transformations

3.1. Every endomorphism T of a space M has an *adjoint* operator U_T acting in the unitary space $L_2(M)$ and defined by the formula

$$U_T f(x) = f(Tx), \quad f \in L_2(M), \ x \in M.$$

U_T is an isometric operator. If T is an automorphism, U_T is unitary; otherwise it is semi-unitary. If U_T and U_{T_1} are operators adjoint to endomorphisms T and T_1 and are isomorphic, then T and T_1 are said to be *spectrally isomorphic*. Properties of an endomorphism that are common to all endomorphisms spectrally isomorphic to it, are called spectral. Clearly, an isomorphism of two endomorphisms implies their spectral isomorphism. The converse is not true; see **16.1**.

The spectral characteristics of U_T are often attributed to T itself. In particular, we talk of the eigenvalues of an endomorphism, of their multiplicities and of automorphisms with a discrete spectrum. Some clarification of the terminology used in connection with the continuous part of the spectrum is needed. Since 1 is an eigenvalue of every U_T (the constants are invariant functions), the spectrum of such an operator can never be continuous. We say that an automorphism T has a continuous spectrum if the operator U_T has a continuous spectrum in the orthogonal complement $L_2(M) \ominus C(M)$ of the subspace of constants. Similarly we define Lebesgue and absolutely continuous spectra for automorphisms.

3.2. An endomorphism T is called *ergodic* if every measurable set A that is invariant under T $(T^{-1}A = A)$ has either measure 1 or measure 0. An equivalent condition: every invariant function in $L_2(M)$ is a constant, that is, 1 is a simple eigenvalue of U_T. Thus, ergodicity is a spectral property.

If T is not ergodic, then it can be decomposed into ergodic components in the following sense. We say that a partition ζ is *fixed* under T if it is measurable and its elements are invariant under T. We denote by T_C the transformation induced by T in an element C of a fixed partition ζ. T_C is an endomorphism of C (with measure μ_C) and is said to be the component of T in C. It can be shown that in the set of all measurable partitions fixed under T there is a finest mod 0 partition and that T is ergodic in the elements of this partition.

3.3. An endomorphism T is said to be *periodic* at a point $x \in M$ if there exists an integer p such that $T^p x = x$. T is said to be *aperiodic* if the set of points of periodicity has measure zero. If the measure in M is continuous, then every ergodic endomorphism is aperiodic.

If T is an aperiodic endomorphism, then for any positive δ there exists a measurable set A of measure less than δ such that a finite number of the sets $T^{-k}A$ cover M.

If T is an aperiodic automorphism, then for any natural number n and any positive δ, there exists a measurable set $A \subset M$ such that the sets $A, TA, \ldots, T^{n-1}A$ are pairwise disjoint and the complement of their union has a measure less than δ.

3.4. We say that the endomorphism T is mixing on the sets $A_0, \ldots, A_r$ if, for any sequence of complexes of non-negative integers

$(k_1^0, \ldots, k_1^r)$, $(k_2^0, \ldots, k_2^r)$, ... satisfying the condition

$$\lim_{n\to\infty} \min_{0<j\leq r} |k_n^j - k_n^i| = \infty$$

the relation

$$\lim_{n\to\infty} \mu\left(\bigcap_{i=0}^{r} T^{-k_n^i} A_i\right) = \prod_{i=0}^{r} \mu(A_i)$$

holds. We say that T is mixing on the bounded functions $f_0, \ldots, f_r$ if for every such subsequence of complexes

$$\lim_{n\to\infty} \left(\prod_{i=0}^{r} U_T^{k_n^i} f_i, 1\right) = \prod_{i=0}^{r} (f_i, 1). \tag{5}$$

Clearly, the endomorphism T is mixing on the sets $A_0, \ldots, A_r$ if and only if it is mixing on the characteristic functions of these-sets. The usual approximation of bounded measurable functions by linear combinations of the characteristic functions of their Lebesgue sets shows that an endomorphism that is mixing on Lebesgue sets of the bounded functions $f_0, \ldots, f_r$ (that is, on any sets $A_0, \ldots, A_r$ such that A_i is a Lebesgue set of f_i) is mixing on the functions $f_0, \ldots, f_r$.

We say that an endomorphism T is *mixing of degree* r if it is mixing on any measurable sets $A_0, \ldots, A_r$ or, equivalently, mixing on any bounded measurable functions $f_0, \ldots, f_r$.

Mixing of degree 1 is simply called *mixing*. We know that it implies ergodicity and that it is a spectral property of an endomorphism. It is not known whether mixing of degree $r > 1$ is a spectral property.

3.5. A measurable partition ζ of M is said to be invariant under the endomorphism T if $T^{-1}\zeta \leqslant \zeta$, and completely invariant if $T^{-1}\zeta = \zeta$. In the set of all invariant measurable partitions finer than a given partition ξ there is a coarsest $\xi^- = \xi_T^-$ defined by the formula $\xi^- = \bigvee_{k=0}^{\infty} T^{-k}\xi$, and the equation $\xi^- = \xi$ is a necessary and sufficient condition for the invariance of ξ. Similarly, if T is an automorphism, the partition

$$\xi_T = \bigvee_{-\infty}^{\infty} T^k \xi$$

is the coarsest completely invariant measurable partition finer than ξ, and the equation $\xi_T = \xi$ is a necessary and sufficient condition for the complete invariance of ξ.

A measurable partition ξ is said to be a *generator* of an endomorphism T if $\xi_T^- = \varepsilon$. A measurable partition is said to be a *two-sided generator* of an automorphism T if $\xi_T = \varepsilon$. ζ is said to be *exhaustive* under an automorphism T if it is invariant and $\bigvee_0^{\infty} T^n\zeta = \varepsilon$.

If ζ is invariant under an endomorphism T, then T induces a *factor-endomorphism* T_ζ in the factor-space M/ζ. This is an automorphism if and only if ζ is completely invariant. Ergodicity of T implies ergodicity of T_ζ. If T is mixing of degree r, then T_ζ is also mixing of degree r.

The partial order in the set of measurable partitions induces a partial order in the set of factor-endomorphisms of an endomorphism T. In particular, if $\zeta_n \nearrow \zeta$, we write $T_{\zeta_n} \nearrow T_\zeta$.

In the set of all partitions that are completely invariant under an endomorphism T there is a finest $\alpha = \alpha(T)$ defined by the formula

$$\alpha = \bigwedge_0^\infty T^{-n}\varepsilon.$$

The corresponding factor-automorphism T_α is the largest factor-automorphism of T.

An endomorphism T_1: $M_1 \to M_1$ is said to be a *homomorphic image* of an endomorphism T: $M \to M$ if there exists a homomorphism S: $M \to M_1$ such that $ST = T_1S$. S is called the *connection*.

Every factor-endomorphism of an endomorphism T is a homomorphic image of T; the connection is the projection $M \to M/\zeta$. Conversely, every homomorphic image of T is isomorphic to one of its factor-endomorphisms.

3.6. As examples, we consider the Bernoulli automorphisms and endomorphisms. Let X be a Lebesgue space and M a direct product, infinite in both directions, of a sequence of copies of X. A point of M is a sequence $\{x_n\}$, $x_n \in X$ $(n = 0, \pm 1, \dots)$. A *Bernoulii automorphism with the set of states* X is the automorphism T defined by $T\{x_n\} = \{y_n\}$, $y_n = x_{n+1}$. If the sequence $\{x_n\}$ is infinite in one direction only $(n = 0, 1, 2, \dots)$, we obtain a *Bernoulli endomorphism*.

We denote by ξ the partition of M defined by the condition: points $\{x_n\}$ and $\{y_n\}$ belong to the same element of the partition if $x_0 = y_0$. Clearly, ξ is a generator in the one-sided case and a two-sided generator in the two-sided case. In both cases the factor space M/ξ is isomorphic to the space of states X.

3.7. An automorphism T' is said to be a *natural extension* of an endomorphism T if T' has an exhaustive partition ζ such that the factor-endomorphism T'_ζ is isomorphic to T. For example, a Bernoulli automorphism is a natural extension of a Bernoulli endomorphism with the same set of states: for ζ we can take the partition ξ^-, where ξ is the two-sided generator of **3.6.**

In [25] it is proved that every endomorphism T has (to within isomorphism) a unique natural extension and that it is ergodic if and only if T is, and is mixing of the same degree as T. The spectrum of the natural extension is determined similarly. By virtue of all this we need only prove the *existence* of the natural extension. Here is the proof.

For a given endomorphism T of a space M we denote by M' the set of sequences

$$(x_0, x_1, \dots), \qquad x_k \in M, \tag{6}$$

such that $Tx_{k+1} = x_k$. For a set $X \subset M$ we denote by X'_n the set of sequences (6) with $x_n \in X$ and we denote by K_n the collection of X'_n for all possible measurable sets $X \subset M$. Clearly, $K_0 \subset K_1 \subset \dots$ and $K = \bigcup K_n$ is a field of sets. We define a function μ' on K by the formula $\mu'(X'_n) = \mu(X)$ and choose a basis Γ in M such that $T^{-1}G \in \Gamma$ for any $G \in \Gamma$. We may assume that M is complete with respect to Γ. We denote by Γ' the system of all sets $G'_n \in K$ corresponding to $G \in \Gamma$. Clearly, Γ' is a basis in M' and the completeness of M with respect to Γ implies that of M' with respect to Γ'. The function μ' is non-negative, finitely-additive and normalized; the completeness of M' with respect to Γ' implies that it is also countably-additive on K. Therefore it can be extended to a Lebesgue measure making M' into a Lebesgue space, and the transformation T' defined by

$$T'(x_0, x_1, \ldots) = (Tx_0, x_0, x_1, \ldots),$$

is clearly an automorphism of M'.

We denote by ζ the partition of M' defined by the condition: sequences $(x_0, x_1, \ldots)$ and $(\overline{x}_0, \overline{x}_1, \ldots)$ belong to the same element of ζ if $x_0 = \overline{x}_0$. The factor-space M'/ζ is canonically isomorphic to M and clearly this isomorphism takes T'_ζ to T. It is obvious also that ζ is exhaustive under T'.

3.8. If T is an endomorphism of M and T' is an endomorphism of M', then the *direct product* $T \times T'$ defined by

$$T \times T'(x, x') = (Tx, T'x'),$$

is an endomorphism of the space $M \times M'$. If T and T' are automorphisms, then $T \times T'$ is also an automorphism. If T and T' are mixing of degree r, then $T \times T'$ is also mixing of degree r.

§4. Entropy of a measurable partition

4.1. Let ξ be a measurable partition of a space M and let $C_1, C_2, \ldots$ be elements of ξ of positive measure. We put

$$H(\xi) = \begin{cases} -\sum_k \mu(C_k) \lg \mu(C_k) & \text{if} \quad \mu(M - \bigcup_k C_k) = 0, \\ +\infty & \text{if} \quad \mu(M - \bigcup_k C_k) > 0 \end{cases} \tag{7}$$

(logarithms are to the base 2). The sum in the first part of (7) can be finite or infinite. $H(\xi)$ is called the *entropy of* ξ.

The entropy $H(\varepsilon)$ of the partition of M into distinct points is sometimes called the entropy of M.

We put, as usual $\lg 0 = -\infty$, $0 \lg 0 = 0$ and we denote by $m(x; \xi)$ the measure of the element of ξ that contains the point $x \in M$. Obviously, (7) can be written in the form

$$H(\xi) = -\int_M \lg m(x; \xi)\, d\mu. \tag{8}$$

PROPERTIES OF ENTROPY

4.2. $H(\xi) \geqslant 0$; $H(\xi) = 0$ *if and only if* $\xi = \nu$.

Obvious.

4.3. *If* $\xi \leqslant \eta$, *then* $H(\xi) \leqslant H(\eta)$. *If* $\xi \leqslant \eta$ *and* $H(\xi) = H(\eta) < \infty$, *then* $\xi = \eta$.

PROOF. If $\xi \leqslant \eta$, then $m(x; \xi) \geqslant m(x; \eta)$ and, by (8), $H(\xi) \leqslant H(\eta)$. If also

$$m(x; \xi) \geqslant m(x; \eta), \quad H(\xi) = H(\eta) < \infty,$$

then (8) implies that $m(x; \xi) = m(x; \eta)$ and from this and the inequality $\xi \leqslant \eta$ it follows that $\xi = \eta$.

4.4. *If* $\xi_n \nearrow \xi$, *then* $H(\xi_n) \nearrow H(\xi)$.

PROOF. If $\xi_n \nearrow \xi$, then $m(x; \xi_n) \searrow m(x; \xi)$ almost everywhere and so, by (8) and the theorem on the integration of monotonic sequences, $H(\xi_n) \nearrow H(\xi)$.

4.5. *If* $\xi_n \searrow \xi$ *and* $H(\xi_1) < \infty$, *then* $H(\xi_n) \searrow H(\xi)$.

PROOF. If $\xi_n \searrow \xi$, then $m(x; \xi_n) \nearrow m(x; \xi)$ almost everywhere, and by (8) and the theorem on the integration of monotonic sequences, $H(\xi_n) \searrow H(\xi)$.

4.6. *The entropy of a measurable partition is the least upper bound of the entropies of finite measurable coarser partitions.*

PROOF. If ξ is a measurable partition and $\xi_1, \xi_2, \ldots$ is an increasing sequence of finite measurable partitions tending to ξ, then $H(\xi_n) \nearrow H(\xi)$. (see 1.3.)

4.7. *The entropy of a measurable partition into n sets is less than or equal to lg n. The entropy is equal to lg n if and only if every element of the partition has measure 1/n.*

PROOF. Let $p_1, \ldots, p_n$ be the measures of the elements of a partition ξ. Since the function

$$\varphi(x) = x \lg x \tag{9}$$

is strictly convex on the half-line $x \geqslant 0$, for any $x_1, \ldots, x_n$ and any non-negative numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\sum \alpha_i = 1,$$

the inequality

$$\varphi\left(\sum \alpha_i x_i\right) \leqslant \sum \alpha_i \varphi(x_i) \tag{10}$$

holds; with equality if and only if all the x_i are equal. Putting Putting $\alpha_i = 1/n$, $x_i = p_i$ $(i = 1, \ldots, n)$, we get

$$-\sum_{i=1}^{n} p_i \lg p_i \leqslant \lg n,$$

with equality if and only if $p_1 = \ldots = p_n = 1/n$.

4.8. *For any measurable partitions* ξ *and* η,

$$H(\xi\eta) \leqslant H(\xi) + H(\eta). \tag{11}$$

If $H(\xi) < \infty$ *and* $H(\eta) < \infty$, *then equality holds if and only if* ξ *and* η *are independent.*

PROOF. If $H(\xi) = \infty$ or $H(\eta) = \infty$, the inequality is trivial. Let $H(\xi) < \infty$ and $H(\eta) < \infty$, and let p_i, q_j and r_{ij} be the measures of the elements of ξ, η, and $\xi\eta$ so that

$$\sum_j r_{ij} = p_i, \quad \sum_i r_{ij} = q_j.$$

We define φ by (9), fix j and rewrite (10) with $\alpha_i = p_i$ and $x_i = r_{ij}/p_i$. Clearly $\sum\limits_i \alpha_i x_i = q_j$ and (10) gives

$$q_j \lg q_j \leqslant \sum_i p_i \frac{r_{ij}}{p_i} \lg \frac{r_{ij}}{p_i} = \sum_i r_{ij} \lg r_{ij} - \sum_i r_{ij} \lg p_i.$$

Summing over j we get (11). Equality occurs if and only if, for each fixed j, x_i does not depend on i. Summation of the equations $r_{ij} = x_i p_i$ over i shows that the last condition is equivalent to the system of equations $q_j = \dfrac{r_{ij}}{p_i}$, that is, ξ and η are independent.

4.9. *For any finite or infinite sequence of measurable partitions* $\xi_1, \xi_2, \ldots$

$$H(\bigvee \xi_k) \leqslant \sum H(\xi_k).$$

When the sequence is finite, this follows from 4.8; when the sequence is infinite, it follows from 4.8 and 4.4

§5. Mean conditional entropy

5.1. If ξ and η are measurable partitions of a space M, then almost every partition ξ_B, $B \in M/\eta$, (see 1.7) has a well-defined entropy $H(\xi_B)$. This is a non-negative measurable function on the factor space M/η, called the *conditional entropy* of ξ with respect to η. Its integral in M/η, finite or infinite, is called the *mean conditional entropy* of ξ with respect to η and is denoted by $H(\xi/\eta)$:

$$H(\xi/\eta) = \int_{M/\eta} H(\xi_B)\, d\mu_\eta. \tag{12}$$

AN EQUIVALENT DEFINITION. Let $B(x)$ be the element of η containing the point $x \in M$. We denote by $m(x;\, \xi/\eta)$ the measure (in $B(x)$) of the element of the partition $\xi_{B(x)}$ containing x (see the definition of $m(x;\, \xi)$ in 4.1.). Then

$$H(\xi/\eta) = -\int_M \lg m(x;\, \xi/\eta)\, d\mu. \tag{13}$$

This formula has the advantage that the domain of integration does not depend on η.

PROOF of (13). By (8)

$$H(\xi_B) = -\int_B \lg m(x;\, \xi_B)\, d\mu_B$$

(B being an element of η), and therefore

$$H(\xi/\eta) = \int_{M/\eta} H(\xi_B)\, d\mu_\eta = -\int_{M/\eta} d\eta_\eta \int_B \lg m(x;\, \xi_B)\, d\mu_B.$$

It remains to note that lg $m(x;\ \zeta_B)$ is the restriction to B of lg $m(x;\ \zeta/\eta)$ (see I.8.).

PROPERTIES OF THE MEAN CONDITIONAL ENTROPY

5.2. $H(\xi/\nu) = H(\xi)$.

Obvious.

5.3. *If* $\zeta \leqslant \eta$, *then* $H(\xi\zeta/\eta) = H(\xi/\eta)$.

PROOF. If B is an element of η, then $(\xi\zeta)_B = \xi_B$.

5.4. $H(\xi/\eta) \geqslant 0$; $H(\xi/\eta) = 0$ *if and only if* $\xi \leqslant \eta$.

Follows from 4.2. For, $\xi \leqslant \eta$ is equivalent to the collection of equations $\xi_B = \nu_B$ $(B \in M/\eta)$.

5.5. *If* $\xi \leqslant \xi'$, *then* $H(\xi/\eta) \leqslant H(\xi'/\eta)$. *If* $\xi \leqslant \xi'$ *and* $H(\xi/\eta) = H(\xi'/\eta) < \infty$, *then* $\xi\eta = \xi'\eta$.

Follows from 4.3. For $\xi\eta = \xi'\eta$ is equivalent to the collection of equations $\xi_B = \xi'_B (B \in M/\eta)$.

5.6. *For any measurable partitions* ξ, η, *and* ζ

$$H(\xi\eta/\zeta) \leqslant H(\xi/\zeta) + H(\eta/\zeta).$$

If $H(\xi/\zeta) < \infty$ *and* $H(\eta/\zeta) < \infty$, *then equality holds if and only if* ξ *and* η *are independent with respect to* ζ (see I.7.).

For according to 4.8,

$$H(\xi_C\eta_C) \leqslant H(\xi_C) + H(\eta_C), \qquad C \in M/\zeta, \tag{14}$$

and to get the required inequality it is sufficient to integrate (14) over M/ζ. The second part of the theorem is a consequence to the second part of Theorem 4.8.

5.7. *If* $\xi_n \nearrow \xi$, *then for any measurable partition* η

$$H(\xi_n/\eta) \nearrow H(\xi/\eta).$$

For by 4.4,

$$H((\xi_n)_B) \nearrow H(\xi_B), \qquad B \in M/\eta,$$

and we need only integrate this over M/η.

5.8. *If* $\xi_n \searrow \xi$, *then for any measurable partition* η *such that* $H(\xi_1/\eta) < \infty$,

$$H(\xi_n/\eta) \searrow H(\xi/\eta).$$

For by 4.5,

$$H((\xi_n)_B) \searrow H(\xi_B), \quad B \in M/\eta,$$

and we need only integrate this over M/η.

5.9. *For any measurable partitions* ξ, η *and* ζ,

$$H(\xi\eta/\zeta) = H(\xi/\zeta) + H(\eta/\xi\zeta).$$

PROOF. If $\zeta = \nu$, the formula becomes

$$H(\xi\eta) = H(\xi) + H(\eta/\xi). \tag{15}$$

We consider this case first.

If the union of the sets of measure 0 in either ξ or η is a set of positive measure, then (15) is obvious (both sides are equal to ∞). We assume that ξ and η are finite or countable and let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be the elements of these partitions. We put $p_i = \mu(A_i)$, $q_j = \mu(B_j)$, $r_{ij} = \mu(A_i \cap B_j)$. From the conditions $\sum_j r_{ij} = p_i$, $\sum_i r_{ij} = q_j$ it follows that

$$-\sum_{i,\,j} r_{ij} \lg r_{ij} = -\sum_i p_i \lg p_i - \sum_i p_i \sum_j \frac{r_{ij}}{p_i} \lg \frac{r_{ij}}{p_i},$$

and this is (15).

Now we take the general case. From what we have proved, we know that

$$H(\xi_C \eta_C) = H(\xi_C) + H(\eta_C/\xi_C), \qquad C \in M/\zeta,$$

and so

$$\int_{M/\zeta} H(\xi_C \eta_C)\, d\mu_\zeta = \int_{M/\zeta} H(\xi_C)\, d\mu_\zeta + \int_{M/\zeta} H(\eta_C/\xi_C)\, d\mu_\zeta.$$

The left-hand side is equal to $H(\xi\eta/\zeta)$; the first term on the right is $H(\xi/\zeta)$, and it remains to show that

$$\int_{M/\zeta} H(\eta_C/\xi_C)\, d\mu_\zeta = H(\eta/\xi\zeta). \tag{16}$$

But this follows from (13) according to which

$$H(\eta_C/\xi_C) = -\int_C \lg m(x;\ \eta_C/\xi_C)\, d\mu_C,$$

$$H(\eta/\xi\zeta) = -\int_M \lg m(x;\ \eta/\xi\zeta)\, d\mu,$$

and from the transitivity of the canonical system of measures (see **1.7**), by which $\lg m(x;\ \eta_C/\xi_C)$ is the restriction to C of the function $\lg m(x;\ \eta/\xi\zeta)$ (see **1.8**).

5.10. *For any measurable partitions ξ, η, and ζ,*

$$H(\xi/\eta\zeta) \leqslant H(\xi/\zeta). \tag{17}$$

If $H(\xi/\zeta) < \infty$, equality holds if and only if ξ and η are independent relative to ζ.

In particular, for any two measurable partitions ξ and η

$$H(\xi/\eta) \leqslant H(\xi),$$

and if $H(\xi) < \infty$, equality holds if and only if ξ and η are independent.

The proof will be given in **5.12**.

5.11. *If $\eta_n \nearrow \eta$ and ξ is a measurable partition such that $H(\xi/\eta_1) < \infty$, then*

$$H(\xi/\eta_n) \searrow H(\xi/\eta).$$

PROOF OF **5.10** *AND* **5.11**. First we show that if $\eta_n \nearrow \eta$ and ξ is a finite measurable partition, $H(\xi/\eta_n) \to H(\xi/\eta)$; next we prove **5.10**, and finally **5.11**.

Let $A_1, \ldots, A_m$ be the elements of ξ and let p_n and p be the projections of M onto the factor-spaces M/η_n and M/η. By **1.9**

$$\mu_{p_n(x)}(A_i \cap p_n(x)) \longrightarrow \mu_{p(x)}(A_i \cap p(x)) \qquad (i = 1, \ldots, m)$$

in mean square, hence in measure. Since

$$H(\xi_{p_n(x)}) = -\sum_i \mu_{p_n(x)}(A_i \cap p_n(x)) \lg \mu_{p_n(x)}(A_i \cap p_n(x)),$$
$$H(\xi_{p(x)}) = -\sum_i \mu_{p(x)}(A_i \cap p(x)) \lg \mu_{p(x)}(A_i \cap p(x)),$$

we have

$$H(\xi_{p_n(x)}) \longrightarrow H(\xi_{p(x)})$$

in measure and

$$H(\xi_{p_n(x)}) \leqslant m \max_{0 \leqslant t \leqslant 1} (-t \lg t).$$

Consequently

$$H(\xi/\eta_n) = \int_M H(\xi_{p_n(x)})\, d\mu \longrightarrow \int_M H(\xi_{p(x)}) d\mu = H(\xi/\eta).$$

Now we prove Theorem 5.10. If $H(\xi/\zeta) = \infty$, then (17) is obvious. If $H(\eta/\zeta) < \infty$, Theorem 5.10 is an obvious consequence of Theorem 5.6 and Theorem 5.9. We consider the case $H(\xi/\zeta) < \infty$, $H(\eta/\zeta) = \infty$.

If ξ is a finite partition, we find an increasing sequence of finite measurable partitions $\eta_1, \eta_2, \ldots$ converging to η and we write

$$H(\xi/\eta_n\zeta) \leqslant H(\xi/\zeta), \tag{18}$$

when we go to the limit, we obtain (17) on the basis of the special case of **5.11** already proved. Equality holds if and only if equality holds in all the inequalities (18), that is, if and only if ξ and η_n are independent with respect to ζ for all n, or if and only if ξ and η are independent with respect to ζ.

If ξ is infinite, we find an increasing sequence of finite measurable partitions $\xi_1, \xi_2, \ldots,$ converging to ξ and write $H(\xi_n/\eta\zeta) \leqslant H(\xi_n/\zeta)$. Proceeding to the limit we obtain (17) on the basis of Theorem 5.7. If $H(\xi/\eta\zeta) = H(\xi/\zeta)$ then, by Theorem 5.9,

$$H(\xi_n/\eta\zeta) = H(\xi/\eta\zeta) - H(\xi/\xi_n\eta\zeta) \geqslant H(\xi/\zeta) - H(\xi/\xi_n\zeta) = H(\xi_n/\zeta).$$

Therefore equality holds in (17) if and only if equality holds in all the inequalities (18), that is, if and only if ξ_n and η are independent with respect to ζ for all n, or, if and only if ξ and η are independent with respect to ζ.

We prove Theorem 5.11 in the general case. Let ξ be an arbitrary measurable partition. By what has already been proved, it is sufficient to establish that for any positive δ there exists a finite measurable partition $\xi_1 \leqslant \xi$ such that for $n = 1, 2, \ldots,$

$$H(\xi/\eta_n) - H(\xi_1/\eta_n) < \delta. \tag{19}$$

Let ξ_1 be a finite measurable partition coarser than ξ, satisfying (19) for $n = 1$ (see **5.7**). By Theorems 5.9 and 5.10,

$$H(\xi/\eta_n) - H(\xi_1/\eta_n) = H(\xi/\xi_1\eta_n) \leqslant H(\xi/\xi_1\eta_1) = H(\xi/\eta_1) - H(\xi_1/\eta_1) < \delta.$$

5.13. *If* $\eta_n \searrow \eta$, *then* $H(\xi/\eta_n) \nearrow H(\xi/\eta)$.

For a finite partition ξ the proof repeats the appropriate part of the proof of Theorem 5.11. Let ξ be an arbitrary measurable partition and l a number less than $H(\xi/\eta)$. If ξ_1 is a finite measurable partition coarser than ξ such that $H(\xi_1/\eta) > l$ (see **5.7**), then $H(\xi_1/\eta_n) \nearrow H(\xi_1/\eta)$, and for sufficiently large n

$$H(\xi/\eta_n) \geqslant H(\xi_1/\eta_n) > l.$$

5.14. *For any measurable partition* η *and any finite or infinite sequence of measurable partitions* $\xi_1, \xi_2, \ldots$,

$$H(\bigvee \xi_k/\eta) \leqslant \sum H(\xi_k/\eta).$$

For by **4.9**

$$H((\bigvee \xi_k)_B) \leqslant \sum H((\xi_k)_B), \qquad B \in M/\eta,$$

and we need only integrate this over M/η.

§6. Spaces of partitions

6.1. We denote by Z the set of measurable partitions with finite entropy and for ξ and η in Z we put

$$\rho(\xi, \eta) = H(\xi/\eta) + H(\eta/\xi).$$

Since

$$H(\xi/\zeta) \leqslant H(\xi\eta/\zeta) = H(\eta/\zeta) + H(\xi/\eta\zeta) \leqslant H(\eta/\zeta) + H(\xi/\eta)$$

and, similarly,

$$H(\zeta/\xi) \leqslant H(\eta/\xi) + H(\zeta/\eta),$$

we have

$$\rho(\xi, \zeta) \leqslant \rho(\xi, \eta) + \rho(\eta, \zeta).$$

It is clear also that $\rho(\xi, \eta) = \rho(\eta, \xi)$, $\rho(\xi, \eta) \geqslant 0$ and $\rho(\xi, \eta) = 0$ if and only if $\xi = \eta$. Thus, if the elements of Z are regarded as *classes* of equal mod 0 measurable partitions, then ρ is a metric in Z.

In this metric Z is a complete separable space.

To prove separability we take a sequence of finite measurable partitions $\xi_1, \xi_2, \ldots$ such that $\xi_n \nearrow \varepsilon$. The set of all partitions coarser than any ξ_n is countable and, by Proposition 6.3 below, it is dense in Z.

Let us prove that Z is complete, that is, let us show that any fundamental sequence $\xi_1, \xi_2, \ldots$ converges in Z. It is sufficient to consider the case $\rho(\xi_n, \xi_{n+p}) < 2^{-n}$ $(p > 0)$; for from any fundamental sequence we can select a subsequence satisfying this condition and a fundamental sequence that contains a convergent subsequence is convergent. We put

$$\xi = \bigwedge_{l=1}^{\infty} \bigvee_{k=l}^{\infty} \xi_k$$

and show that $\xi \in Z$ and $\rho(\xi, \xi_n) \to 0$. According to 5.9, for $l > n$

$$H\left(\bigvee_{k=l}^{\infty} \xi_k / \bigvee_{k=n}^{l-1} \xi_k\right) = H\left(\xi_l / \bigvee_{k=n}^{l-1} \xi_k\right) + H\left(\bigvee_{k=l+1}^{\infty} \xi_k / \bigvee_{k=n}^{l} \xi_k\right).$$

Summing over l we get

$$H\left(\bigvee_{k=n+1}^{\infty} \xi_k / \xi_n\right) = \sum_{l=n+1}^{\infty} H\left(\xi_l / \bigvee_{k=n}^{l-1} \xi_k\right) \leqslant \sum_{l=n+1}^{\infty} H(\xi_l / \xi_{l-1}),$$

and since

$$\xi \leqslant \bigvee_{k=n+1}^{\infty} \xi_k, \qquad H(\xi_l/\xi_{l-1}) \leqslant \rho(\xi_{l-1}, \xi_l) < 2^{-(l-1)},$$

we have

$$H(\xi/\xi_n) < \sum_{l=n+1}^{\infty} 2^{-(l-1)} = 2^{-(n-1)}.$$

On the other hand, by 5.13,

$$H(\xi_n/\xi) = \lim_{l\to\infty} H\left(\xi_n / \bigvee_{k=l}^{\infty} \xi_k\right),$$

and so, for sufficiently large l,

$$H(\xi_n/\xi) \leqslant H\left(\xi_n / \bigvee_{k=l}^{\infty} \xi_k\right) + 2^{-n} \leqslant H(\xi_n/\xi_l) + 2^{-n}.$$

If $l > n$, then $H(\xi_n/\xi_l) \leqslant \rho(\xi_n, \xi_l) < 2^{-n}$ and we get

$$H(\xi) \leqslant H(\xi_1) + H(\xi/\xi_1) < H(\xi_1) + 1 < \infty,$$
$$\rho(\xi, \xi_n) = H(\xi/\xi_n) + H(\xi_n/\xi) < 2^{-(n-2)}.$$

6.2. *The set of finite partitions is dense in Z.*

For if $\xi \in Z$ and $\xi_1, \xi_2, \ldots$ is a sequence of finite partitions such that $\xi_n \nearrow \xi$, then

$$\rho(\xi_n, \xi) = H(\xi/\xi_n) \to H(\xi/\xi) = 0.$$

6.3. *Let $\xi_1, \xi_2, \ldots$ be a sequence of measurable partitions. If then the set of partitions $\xi \in Z$ such that $\xi \leqslant \xi_n$ for at least one n is dense in Z.*

It is sufficient to prove that for any finite partition $\eta \in Z$ and any positive δ there exists an n and a $\xi \in Z$ such that $\xi \leqslant \xi_n$ and $\rho(\xi, \eta) < \delta$.

Let $C_1, \ldots, C_m$ be the elements of η. As $\xi_n \nearrow \varepsilon$, we have $\mathfrak{M}(\xi_n) \nearrow \mathfrak{M}(\xi)$. Consequently, for any positive δ' there exists an n and ξ_n-sets $C_1', \ldots, C_{m-1}'$ such that $\rho(C_i, C_i') < \delta'$ $(i = 1, \ldots, m-1)$. We denote by ξ the partition of M into sets $D_1, \ldots, D_m$ defined by the formulae

$$D_1 = C_1', \quad D_i = C_i' - \bigcup_{j=1}^{i-1} C_j' \quad (i = 2, \ldots, m-1),$$
$$D_m = M - \bigcup_{j=1}^{m-1} C_j'.$$

Clearly, $\xi \leqslant \xi_n$ and

$$\rho(\xi, \eta) = \sum_{i=1}^{m} \mu(C_i) \lg \mu(C_i) + \sum_{i=1}^{m} \mu(D_i) \lg \mu(D_i) - $$
$$- 2 \sum_{i, j=1}^{m} \mu(C_i \cap D_j) \lg \mu(C_i \cap D_j).$$

These formulae show that $\rho(\xi, \eta)$ depends continuously on $C_1', \ldots, C_{m-1}'$ (see 1.4) and is zero for $C_1' = C_1, \ldots, C_{m-1}' = C_{m-1}$. Therefore, if δ' is sufficiently small, then $\rho(\xi, \eta) < \delta$.

6.4. *If $\xi_n \nearrow \xi$, then the set of partitions in Z coarser than some ξ_n is dense in the set of all partitions in Z coarser than ξ.*

This theorem reduced to Theorem 6.3 (of which it is a generalization) after factoring M by ξ.

6.5. *The function $H(\xi)$ is continuous on Z; the function $H(\xi/\eta)$ is continuous on $Z \times Z$. Moreover, for any three partitions ξ, η, ζ in Z,*

$$|H(\xi/\zeta) - H(\eta/\zeta)| \leqslant \rho(\xi, \eta),$$
$$|H(\xi/\eta) - H(\xi/\zeta)| \leqslant \rho(\eta, \zeta).$$

In fact,

$$H(\xi/\zeta) - H(\eta/\zeta) \leqslant H(\xi\eta/\zeta) - H(\eta/\zeta) = H(\xi/\eta\zeta) \leqslant H(\xi/\eta),$$
$$H(\xi/\eta) - H(\xi/\zeta) \leqslant H(\xi\zeta/\eta) - H(\xi/\eta\zeta) = H(\zeta/\eta).$$

6.6. *If A is an everywhere dense set in Z and ξ, η are measurable partitions such that*

$$H(\alpha/\xi) = H(\alpha/\eta) \tag{20}$$

for any $\alpha \in A$, then $\xi = \eta$.

PROOF. If (20) holds for any $\alpha \in A$, then by Theorem 6.5 it is true for every $\alpha \in Z$, and then by Theorem 5.7, it is true for every measurable α. Putting $\alpha = \xi$ first and then $\alpha = \eta$, we see that $\xi \leqslant \eta$ and $\eta \leqslant \xi$.

6.7. The set Z_1 of all measurable partitions also has a natural topology. This can be described in many equivalent forms; here is one of them.

We denote by Ω the class of subsets of Z_1 each one of which is defined by finitely many inequalities of the form $|H(\alpha/\xi) - H(\alpha/\xi_0)| < \delta$, where $\xi_0 \in Z_1$, $\alpha \in Z$, and δ is a positive number. Clearly the sets of Ω cover Z_1, and if the intersection of two sets of Ω contains a partition ξ_1, then this intersection contains a set of Ω that contains ξ_1. Consequently Ω is a basis of a topology in Z_1.

This is a Hausdorff topology. For if $\xi_0 \neq \xi_1$, then by **6.6**, there exists a partition $\alpha \in Z$ such that $H(\alpha/\xi_0) \neq H(\alpha/\xi_1)$; putting

$\delta = \frac{1}{2}|H(\alpha/\xi_0) - H(\alpha/\xi_1)|$ we see that the neighbourhoods of ξ_0 and ξ_1 defined by

$$|H(\alpha/\xi) - H(\alpha/\xi_0)| < \delta, \quad |H(\alpha/\xi) - H(\alpha/\xi_1)| < \delta$$

do not intersect.

From Theorem 6.5 it follows that the identity transformation from Z to Z_1 is continuous and, from Theorem 5.11, that Z is dense in Z_1. Thus, any set dense in Z is dense in Z_1 and so Z_1 is separable. It can be shown that it has a metric in which it is complete.

§7. Fundamental Lemmas

7.1. Let T be an endomorphism and ξ a measurable partition of the space M. We put

$$h(T, \xi) = H(\xi/T^{-1}\xi^-). \tag{21}$$

If ξ is an invariant partition, then $\xi^- = \xi$ and the formula simplifies to $h(T, \xi) = H(\xi/T^{-1}\xi)$. It is also clear that $h(T, \xi) = H(T, \xi^-)$ for every measurable partition ξ so that the function $h(T, \xi)$ attains all its values on the set of invariant partitions.

The properties of $h(T, \xi)$ will be studied in the next section. This section contains subsidiary material. We use the following notation:

$$\xi_T^n = \bigvee_0^{n-1} T^{-k}\xi, \quad n \geqslant 1; \quad \xi_T^0 = \nu.$$

Simpler notation: ξ^n.

7.2. *If $\eta \leqslant \xi$, then*

$$H(\xi^n/T^{-n}\eta^-) = \sum_{k=0}^{n-1} H(\xi/T^{-1}(\eta^-\xi^k)). \tag{22}$$

In particular,

$$H(\xi^n/T^{-n}\xi^-) = nh(T, \xi).$$

PROOF. Since $\xi^k = \xi \cdot T^{-1}\xi^{k-1}$, we have

$$H(\xi^k/T^{-k}\eta^-) = H(T^{-1}\xi^{k-1}/T^{-k}\eta^-) + H(\xi/T^{-k}\eta^- T^{-1}\xi^{k-1})$$

(see **5.9**). As T is an endomorphism and $\eta < \xi$,

$$H(T^{-1}\xi^{k-1}/T^{-k}\eta^-) = H(\xi^{k-1}/T^{-(k-1)}\eta^-),$$
$$T^{-k}\eta^- T^{-1}\xi^{k-1} = T^{-1}(\eta^-\xi^{k-1})$$

and so

$$H(\xi^k/T^{-k}\eta^-) = H(\xi^{k-1}/T^{-(k-1)}\eta^-) + H(\xi/T^{-1}(\eta^-\xi^{k-1})).$$

(22) is derived from this equation by an obvious induction.

7.3. *If $\eta \leqslant \xi$ and $H(\xi/T^{-1}\eta^-) < \infty$, then*

$$\frac{1}{n} H(\xi^n/T^{-n}\eta^-) \searrow h(T, \xi). \tag{23}$$

PROOF. Since $\xi^n \nearrow \xi^-$, we have $\eta^-\xi^n \nearrow \xi^-$ and

$$H(\xi/T^{-1}(\eta^-\xi^n)) \searrow H(\xi/T^{-1}\xi^-) = h(T, \xi). \tag{24}$$

(23) follows from (22), (24) and the standard theorem on arithmetic means.

7.4. *If* $\xi \in Z$, *then*

$$\frac{1}{n} H(\xi^n) \searrow h(T, \xi). \tag{25}$$

This is a special case of the preceding theorem: put $\eta = \nu$.

7.5. *If* $\xi \leqslant \eta$ *and* $H(\eta/T^{-1}\xi^-) < \infty$, *then*

$$\frac{1}{n} H(\xi^n/T^{-n}\eta^-) \to h(T, \xi).$$

PROOF. Let δ be a positive number. Since $\frac{1}{n} H(\xi^n/T^{-n}\eta^-) \leqslant \frac{1}{n} H(\xi^n/T^{-n}\xi^-) = h(T, \xi)$ (see **7.2**) and since by Theorem 7.3

$\frac{1}{n} H(\eta^n/T^{-n}\xi^-) \searrow h(T, \eta)$, it is sufficient to prove that the inequality

$\frac{1}{n} H(\xi^n/T^{-n}\eta^-) > h(T, \xi) - \delta$ holds for all n for which

$\frac{1}{n} H(\eta^n/T^{-n}\xi^-) < h(T, \eta) + \delta$. This is clear from the chain of relations:

$$\begin{aligned}\frac{1}{n} H(\xi^n/T^{-n}\eta^-) &= \frac{1}{n} H(\eta^n/T^{-n}\eta^-) - \frac{1}{n} H(\eta^n/\xi^n T^{-n}\eta^-) \geqslant \\ &\geqslant h(T, \eta) - \frac{1}{n} H(\eta^n/\xi^n T^{-n}\xi^-) \geqslant \\ &\geqslant \frac{1}{n} H(\eta^n/T^{-n}\xi^-) - \delta - \frac{1}{n} H(\eta^n/\xi^n T^{-n}\xi^-) = \\ &= \frac{1}{n} H(\xi^n/T^{-n}\xi^-) - \delta = h(T, \xi) - \delta.\end{aligned}$$

7.6. *If* ξ, η, ζ *are measurable partitions such that* $\xi \leqslant \eta$ *and* $H(\eta\zeta/T^{-1}\eta^-) < \infty$, *then*

$$H(\xi/T^{-1}\eta^- \cdot T^{-n}\zeta^-) \nearrow H(\xi/T^{-1}\eta^-). \tag{26}$$

The proof depends on the formula

$$H(\eta^n/T^{-n}(\eta^-\zeta^-)) = \sum_{k=1}^{n} H(\eta/T^{-1}\eta^- \cdot T^{-k}\zeta^-), \tag{27}$$

which is true for all measurable partitions η and ζ and can be derived by induction from the equation

$$\begin{aligned}&H(\eta^k/T^{-k}\eta^- \cdot T^{-n}\zeta^-) = \\ &= H(T^{-(k-1)}\eta/T^{-k}\eta^- \cdot T^{-n}\zeta^-) + H(\eta^{k-1}/T^{-(k-1)}\eta \cdot T^{-k}\eta^- \cdot T^{-n}\zeta^-) = \\ &= H(\eta/T^{-1}\eta^- \cdot T^{-(n+1-k)}\zeta^-) + H(\eta^{k-1}/T^{-(k-1)}\eta^- T^{-n}\zeta^-).\end{aligned}$$

If $\xi = \eta$, then (26) is a consequence of (27) and of Theorem 7.5. For the sequence $\{H(\eta/T^{-1}\eta^- \cdot T^{-n}\zeta^-)\}$ is monotonic and therefore converges, and by (27), the theorem on arithmetic means and Theorem 7.5, its limit is $H(\eta/T^{-1}\eta^{-})$.

In the general case

$$H(\xi/T^{-1}\eta^{-}\cdot T^{-n}\zeta^{-}) = H(\eta/T^{-1}\eta^{-}\cdot T^{-n}\zeta^{-}) - H(\eta/\xi T^{-1}\eta^{-}\cdot T^{-n}\zeta^{-}).$$

The first term on the right, according to what we have just proved, has the limit $H(\eta/T^{-1}\eta^{-})$; the second term does not exceed $H(\eta/\xi\cdot T^{-1}\eta^{-})$. Consequently

$$\lim H(\xi/T^{-1}\eta^{-}\cdot T^{-n}\zeta^{-}) \geqslant H(\eta/T^{-1}\eta^{-}) - H(\eta/\xi\cdot T^{-1}\eta^{-}) = H(\xi/T^{-1}\eta^{-}),$$

and the opposite inequality is obvious.

7.7. *If T is an automorphism and ξ, η are measurable partitions such that $H(\xi\eta/T^{-1}\xi^{-}) < \infty$, then*

$$h(T, \xi\eta) - h(T, \xi) = H(\eta/T^{-1}\eta^{-}\cdot\xi_T). \tag{28}$$

The proof depends on the formula

$$H(\eta^n/\xi^{-}\cdot T^{-n}\eta^{-}) = \sum_{k=0}^{n} H(\eta/T^{-1}\eta^{-}\cdot T^k\xi^{-}), \tag{29}$$

which holds for any measurable partitions ξ and η and is derived by induction from the equation

$$\begin{aligned} H(\eta^k/\xi^{-}\cdot T^{-k}\eta^{-}) &= H(T^{-(k-1)}\eta/\xi^{-}\cdot T^{-k}\eta^{-}) + H(\eta^{k-1}/T^{-(k-1)}\eta^{-}\xi^{-}) = \\ &= H(\eta/T^{k-1}\xi^{-}\cdot T^{-1}\eta^{-}) + H(\eta^{k-1}/\xi^{-}T^{-(k-1)}\eta^{-}). \end{aligned} \tag{30}$$

(28) follows from (29) and Theorem 7.5. For

$$\begin{aligned} \frac{1}{n}[H(\xi^n\eta^n/T^{-n}(\xi^{-}\eta^{-})) - H(\xi^n/T^{-n}(\xi^{-}\eta^{-}))] &= \\ = \frac{1}{n} H(\eta^n/\xi^n T^{-n}\xi^{-}\cdot T^{-n}\eta^{-}) &= \frac{1}{n} H(\eta^n/\xi^{-}\cdot T^{-n}\eta^{-}). \end{aligned} \tag{31}$$

By Theorem 7.5 the left-hand side of this equation converges to the left-hand side of (28). By (29), the theorem on arithmetic means and the fact that

$$H(\eta/T^{-1}\eta^{-}\cdot T^n\xi^{-}) \searrow H(\eta/T^{-1}\eta^{-}\cdot\xi_T)$$

the right-hand side of (31) converges to the right-hand side of (28).

§8. Properties of the function $h(T, \xi)$

8.1. *$h(T, \xi) \leqslant H(\xi)$. In particular,* if *$\xi \in Z$, then $h(T, \xi) < \infty$.*

This follows directly from the definition of $h(T, \xi)$ and Theorem 5.10.

8.2. *$h(T, \xi\eta) \leqslant h(T, \xi) + h(T, \eta)$. If ξ^{-} and η^{-} are independent, then equality holds.*

PROOF. By Theorem 5.9,

$$H(\xi\eta/T^{-1}\xi^{-}\cdot T^{-1}\eta^{-}) = H(\xi/T^{-1}\xi^{-}\cdot T^{-1}\eta^{-}) + H(\eta/\xi^{-}\cdot T^{-1}\eta^{-}),$$

and by Theorem 5.10

$$H(\xi/T^{-1}\xi^{-}\cdot T^{-1}\eta^{-}) \leqslant H(\xi/T^{-1}\xi^{-}), \quad H(\eta/\xi^{-}T^{-1}\eta^{-}) \leqslant H(\eta/T^{-1}\eta^{-}). \tag{32}$$

It remains to note that if ξ^- and η^- are independent, then equality holds in the inequalities (32).

8.3. *If $n \geqslant 1$, then $h(T, \xi) \leqslant h(T, \xi^n)$*

For since $(\xi^n)^- = \xi^-$, we have

$$H(\xi^n/T^{-1}(\xi^n)^-) = H(\xi^n/T^{-1}\xi^-) \geqslant H(\xi/T^{-1}\xi^-).$$

8.4. $h(T^n, \xi^n) = nh(T, \xi)$.

PROOF. Since $(\xi^n_T)^-_{T^n} = \xi^-_T$, we have

$$h(T^n, \xi^n_T) = H(\xi^n_T/T^{-n}(\xi^n_T)^-_{T^n}) = H(\xi^n_T/T^{-1}\xi^-_T),$$

and it remains to apply Theorem 7.2.

8.5. *If T is an automorphism and $\xi \in Z$, then $h(T^{-1}, \xi) = h(T, \xi)$.*

This follows from Theorem 7.4 and the obvious relation $\xi^n_{T^{-1}} = T^{n-1}\xi^n_T$.

NOTE. The condition $\xi \in Z$ is necessary. If, for example, the partition ξ is invariant under T, then $h(T^{-1}, \xi) = H(\xi/T\xi) = 0$, but there exists automorphisms for which $h(T, \xi)$ is not identically equal to zero.

8.6. *The function $h(T, \xi)$ is continuous on Z (in ξ). Moreover, for any $\xi \in Z$, $\eta \in Z$,*

$$|h(T, \eta) - h(T, \xi)| \leqslant \rho(\xi, \eta). \tag{33}$$

PROOF. Since

$$H(\xi^n) + H(\eta^n/\xi^n) = H(\xi^n\eta^n) = H(\eta^n) + H(\xi^n/\eta^n),$$

we have

$$H(\eta^n) - H(\xi^n) = H(\eta^n/\xi^n) - H(\xi^n/\eta^n),$$

and

$$|H(\eta^n) - H(\xi^n)| \leqslant H(\eta^n/\xi^n) + H(\xi^n/\eta^n). \tag{34}$$

But

$$H(\xi^n/\eta^n) \leqslant \sum_{k=0}^{n-1} H(T^{-k}\xi/\eta^n) \leqslant \sum_{k=0}^{n-1} H(T^{-k}\xi/T^{-k}\eta) = nH(\xi/\eta) \tag{35}$$

and similarly,

$$H(\eta^n/\xi^n) \leqslant nH(\eta/\xi). \tag{36}$$

From (34)-(36) it follows that $|H(\eta^n) - H(\xi^n)| \leqslant n\rho(\xi, \eta)$. Dividing by n and taking the limit as $n \to \infty$, we obtain (33).

8.7. *If $\xi \leqslant \eta$ and $H(\eta/T^{-1}\xi^-) < \infty$, then*

$$h(T, \xi) \leqslant h(T, \eta).$$

In particular, $h(T, \xi)$ is monotonic on Z.

PROOF. Since $\xi \leqslant \eta$, we have

$$\frac{1}{n} H(\xi^n/T^{-n}\eta^-) \leqslant \frac{1}{n} H(\eta^n/T^{-n}\eta^-).$$

But by Theorem 7.2 the right-hand side is equal to $h(T, \eta)$ and by Theorem 7.5 the limit of the left-hand side (as $n \to \infty$) is $h(T, \xi)$.

NOTE. On the set of *all* measurable partitions the function $h(T, \xi)$ is not monotonic, in general, For example. if T is an automorphism, then clearly $h(T, \varepsilon) = 0$, but as has already been said, there exists an automorphism for which $h(T, \xi)$ is not identically zero.

8.8. *If $\xi \leqslant \eta^-$ and $\eta \in Z$, then*

$$h(T, \xi) \leqslant h(T, \eta). \tag{37}$$

PROOF. Since $\eta^m \nearrow \eta^-$, the set of partitions coarser then the partitions η^m is dense in the set of measurable partitions coarser than η^- (see Theorem 6.4). Therefore it is sufficient to prove (37) for the case $\xi \leqslant \eta^m$ for some m. But in this case $\xi^n \leqslant (\eta^m)^n = \eta^{m+n-1}$ and

$$h(T, \xi) = \lim \frac{1}{n} H(\xi^n) \leqslant \lim \frac{1}{n} H(\eta^{m+n-1}) = h(T, \eta).$$

8.9. *If T is an automorphism and $\xi \leqslant \eta_T$, $\eta \in Z$, then $h(T, \xi) \leqslant h(T, \eta)$.*

The proof is similar to the preceding one. Since $\eta^m \cdot T^{m-1}\eta^m \nearrow \eta_T$, it is sufficient to consider the case $\xi \leqslant \eta^m \cdot T^{m-1}\eta^m$ for some m. In this case $\xi^n \leqslant (\eta^m T^{m-1}\eta^m)^n = T^{m-1}\eta^{2m+n-2}$ and

$$h(T, \xi) = \lim \frac{1}{n} H(\xi^n) \leqslant \lim \frac{1}{n} H(\eta^{2m+n-2}) = h(T, \eta).$$

8.10. *If $\xi \in Z$ and η is a partition fixed with respect to an endomorphism T, then*

$$h(T, \xi\eta) = h(T, \xi).$$

PROOF. First we assume that $\eta \in Z$ and write down (31) again. Since a fixed partition is completely invariant, the right-hand side of this equation is zero; the left-hand side by Theorem 7.5 converges to $h(T, \xi\eta) - h(T, \xi)$. Consequently, $h(T, \xi\eta) = h(T, \xi)$.

If η is an arbitrary fixed partition, then there exists a sequence $\eta_1, \eta_2, \ldots$ in Z such that $\eta_n \nearrow \eta$, and, as any partition coarser than a fixed partition is fixed, $h(T, \xi/\eta_n) = h(T, \xi)$. Since η and η_n are completely invariant,

$$h(T, \xi\eta_n) = H(\xi/\eta_n T^{-1}\xi^-), \quad h(T, \xi\eta) = H(\xi/\eta T^{-1}\xi^-),$$

and by applying Theorem 5.11 we see that $h(T, \xi\eta_n) \to h(T, \xi\eta)$. Consequently, $h(T, \xi\eta) = h(T, \xi)$.

8.11. *If $\xi \in Z$ and η is a partition fixed under T, then*

$$h(T, \xi) = \int_{M/\eta} h(T_B, \xi_B)\, d\mu_\eta, \tag{38}$$

where T_B is the component of the endomorphism T in the element B of η.

PROOF. Since

$$h(T_B, \xi_B) = H(\xi_B/T_B^{-1}\xi_B^-) = -\int_B \lg m(x; \xi_B/T_B^{-1}\xi_B^-)\, d\mu_B$$

(see 5.1) and since the function under the integral sign is the restriction to B of $\lg m(x; \xi/\eta \cdot T^{-1}\xi^-)$, we have

$$\int\limits_{M/\eta} h(T_B, \xi_B)\, d\mu_\eta = -\int\limits_{M/\eta} d\mu_\eta \int\limits_B \lg m(x;\ \xi_B/T_B^{-1}\xi_B^-)\, d\mu_B = \\ = -\int\limits_M \lg m(x;\ \xi/\eta T^{-1}\xi^-)\, d\mu = H(\xi/\eta T^{-1}\xi^-), \quad (39)$$

The right-hand side is equal to $h(T,\ \xi\eta) = h(T,\ \xi)$ (see **8.10**).

§9. Entropy of an endomorphism

For an arbitrary endomorphism T of M we put

$$h(T) = \sup h(T,\ \xi), \quad (40)$$

where the upper bound is taken over all measurable partitions or (what by **7.1**, gives the same result) over all invariant partitions. $h(T)$ is called the *entropy of the endomorphism* T. It is a non-negative number or $+\infty$.

The right-hand side of (40) *does not change if the upper bound is taken only over Z or even only over the set of finite measurable partitions.*

PROOF. It is sufficient to show that for any measurable ξ and any positive number $l < h(T,\ \xi)$ there exists a finite measurable partition η such that $h(T,\ \eta) > l$.

Let $\xi_1, \xi_2, \ldots$ be a sequence of finite measurable partitions such that $\xi_n \nearrow \xi$. Since $\xi_n \leqslant \xi$, we have

$$h(T,\ \xi_n) = H(\xi_n/T^{-1}\xi_n^-) \geqslant H(\xi_n/T^{-1}\xi^-).$$

The right-hand side of this inequality converges to $H(\xi/T^{-1}\xi^-) = h(T,\ \xi)$. Consequently $h(T,\ \xi_n) > l$ for sufficiently large n and we can put $\eta = \xi_n$.

9.2. *If S is a factor-endomorphism of an endomorphism T, then* $h(S) \leqslant h(T)$.

PROOF. For $h(T)$ is the least upper bound of the function $h(T,\ \xi)$ on the set of all measurable partitions and $h(S)$ is the least upper bound of the same function on the set of measurable partitions coarser than ζ for which $S = T_\zeta$.

9.3. *For any endomorphism T and any $n \geqslant 0$, $h(T^n) = nh(T)$. If T is an automorphism, then $h(T^{-1}) = h(T)$.*

PROOF. By **8.3** and **9.1**, $h(T^n,\ \xi) \leqslant h(T^n,\ \xi^n) \leqslant h(T^n)$ for any measurable partition ξ. The least upper bound of the left-hand side (over ξ) is equal to $h(T^n)$ and hence this is also the least upper bound of $h(T^n,\ \xi^n)$ over ξ. But according to **8.4**, $h(T^n,\ \xi^n) = nh(T,\ \xi)$ and so $\sup h(T^n,\ \xi^n) = n \sup h(T,\ \xi) = nh(T)$. The second part of the theorem, concerning automorphisms, follows from Theorem 8.5.

9.4. *If $\xi \in Z$ is a generator for an endomorphism T or a two-sided generator for an automorphism* (see **3.5**), *then $h(T,\ \xi) = h(T)$.*

This follows from Theorems 8.8 and 8.9.

9.5. *If $\xi_1, \xi_2, \ldots$ is a sequence of partitions in Z such that $\xi_n \nearrow \varepsilon$, then $h(T,\ \xi_n) \nearrow h(T)$.*

PROOF. Let l be an arbitrary number less than $h(T)$. We look for a partition $\xi \in Z$ such that $h(T,\ \xi) > l$ and for an $\eta \in Z$, and an n such that

$\eta \leqslant \xi_n$ and $\rho(\xi, \eta) < h(T, \xi) - l$ (see **6.3**). According to **8.6**, $h(T, \xi) - h(T, \eta) < \rho(\xi, \eta)$, and hence $h(T. \xi_n) \geqslant h(T, \eta) > l$. Thus, if $l < h(T)$, there exists an n such that $h(T, \xi_n) > l$. Since $h(T, \xi_n) \leqslant h(T)$ and the sequence $h(T, \xi_1)$, $h(T, \xi_2)$, ... is increasing, we see that $h(T, \xi_n) \nearrow h(T)$.

9.6. *If* $T_{\xi_n} \nearrow T$ (see **3.6**), *then* $h(T_{\xi_n}) \nearrow h(T)$.

PROOF. If $\xi_n \nearrow \varepsilon$, then there exist $\eta_n \in Z$ such that $\eta_n \leqslant \xi_n$ and $\eta_n \nearrow \varepsilon$. The relation $h(T_\xi) \nearrow h(T)$ follows from the inequalities $h(T, \eta_n) \leqslant h(T_{\xi_n}) \leqslant h(T)$ and Theorem **9.5**.

9.7. *For any two endomorphisms S and T,*

$$h(S \times T) = h(S) + h(T) \tag{41}$$

(see **3.8**).

PROOF. Let X and Y be the spaces in which S and T act, and let ν_X, ν_Y, ε_X, ε_Y be the trivial and the point partitions of these spaces. Also, let $\xi_1, \xi_2, \ldots$ be a sequence of partitions in X such that $\xi_n \nearrow \varepsilon_X$ and $\eta_1, \eta_2, \ldots$. a sequence of partitions in Y such that $\eta_n \nearrow \varepsilon_Y$. It is clear that the partitions $\xi_n \times \nu_Y$ and $\nu_X \times \eta_n$ are independent and that $(\xi_n \times \nu_Y)(\nu_X \times \eta_n) = \xi_n \times \eta_n \nearrow \varepsilon_X \times \varepsilon_Y$. Therefore

$$h(S \times T, \xi_n \times \eta_n) = h(S \times T, \xi_n \times \nu_Y) + h(S \times T, \nu_X \times \eta_n) = h(S, \xi_n) + h(T, \eta_n)$$

(see **8.2**). (41) follows from this equation and Theorem 9.5.

9.8. *If a partition η is fixed under an endomorphism T, then*

$$h(T) = \int_{M/\eta} h(T_B)\, d\mu_\eta,$$

where T_B is the component of T in the element B of η.

PROOF. Let $\xi_1, \xi_2, \ldots$ be a sequence in Z such $\xi_n \nearrow \varepsilon$. According to Theorem 8.11,

$$h(T, \xi_n) = \int_{M/\eta} h(T_B, (\xi_n)_B)\, d\mu_\eta, \tag{42}$$

and according to Theorem 9.6,

$$h(T, \xi_n) \nearrow h(T), \ h(T_B, (\xi_n)_B) \nearrow h(T_B). \tag{43}$$

The required equation follows from (42), (43), and the theorem on the integration of monotonic sequences.

9.9. *The entropy of an endomorphism is equal to the entropy of its natural extension.*

This follows from Theorem 9.6: if ζ is exhaustive under the automorphism T, then $T_{T^n\zeta} \nearrow T$, and the factor-endomorphisms are isomorphic to T_ζ.

9.10. In the simplest cases the entropy of an endomorphism can be computed directly.

If T is the identity automorphism, then every measurable partition is completely invariant so that $h(T, \xi)$ is identically zero and $h(T) = 0$.

If T is a periodic automorphism, then T^p is the identity automorphism for some p and since $h(T) = h(T^p)/p$ (see **9.3**), $h(T) = 0$.

If T is a Bernoulli automorphism or endomorphism with a space of states X, then $h(T)$ is equal to the entropy of X.

PROOF. Let ξ be the generator defined in **3.6**. As X is isomorphic to the factor-space M/ξ, its entropy is equal to $H(\xi)$. So we have to prove that $h(T) = H(\xi)$.

If $H(\xi) < \infty$, then by Theorem 9.4 $h(T) = h(T, \xi)$, and the independence of ξ and $T^{-1}\xi^-$ gives

$$h(T, \xi) = H(\xi / T^{-1}\xi^-) = H(\xi).$$

Therefore, in this case, $h(T) = H(\xi)$.

If $H(\xi) = \infty$, then there are partitions coarser than ξ having arbitrarily large finite entropy and, clearly, for one of these, say η, the factor-endomorphism T_{η^-} is isomorphic to a Bernoulli endomorphism with space of states isomorphic to M/η. Consequently, if $H(\xi) = \infty$, then T has factor-endomorphisms with arbitrarily large entropy and so $h(T) = \infty$ (see **9.2**).

§10. The existence of generations

10.1. If an automorphism T has a two-sided generator $\xi \in Z$ then by **9.4** and **8.1**, $h(T) = h(T, \xi) \leqslant H(\xi)$. In particular, an automorphism having a two-sided generator in Z, has finite entropy.

Another necessary condition for the existence of a two-sided generator when M has continuous measure is that the automorphism must be *aperiodic*. This condition is necessary for the existence of a *countable* two-sided generator; for if an automorphism T is periodic on a set A and the partition ξ is finite or countable, then the partition ξ_T induces in A only a finite or countable partition.

Now we turn to endomorphisms. If an endomorphism T has a generator with finite entropy, then by **9.4** and **8.1**,

$$h(T) = h(T, \varepsilon) < \infty.$$

Furthermore, the existence of a finite or countable generator clearly implies that the inverse image of points (that is, the elements of the partition $T^{-1}\varepsilon$) is finite or countable, and if the measure on M is continuous, that T is aperiodic.

The main results of this section are the converses of these propositions (see **10.7**, **10.11** and **10.13**). As we shall see, the greatest difficulty is the existence of generators with finite entropy. The existence of a finite or countable generator, for an aperiodic endomorphism with a finite or countable inverse image of points, is easily proved (see **10.13**) and does not involve entropy theory. However, it is an important fact; it implies, for example, that every aperiodic automorphism (to within isomorphism) generates a stationary process with a countable number of states (discrete time).

The theorem on the existence of a two-sided generator in Z for an aperiodic automorphism with finite entropy (see **10.7**) is of historical as well as factual interest. The fact of the matter is that the original entropy characteristic of an automorphism T, which was proposed by Kolmogorov [14] and denoted by him as $h_1(T)$, was defined as $h(T, \xi)$ where ξ is a two-sided generator with finite entropy, provided that such a generator exists, and as ∞ otherwise. Since this did not explain for what automorphisms such generators do or do not exist, Sinai [30] proposed to change Kolmogorov's invariant to the now generally used entropy $h(T)$. Theorem 10.7 shows that, for aperiodic automorphisms $h(T) = h_1(T)$.

There are unsolved problems connected with generators. For example, what automorphisms have finite two-sided generators? or m-term generators?

LEMMAS ON PARTITIONS

10.2. *For any two measurable partitions α and β such that $\alpha \geqslant \beta$ and $H(\alpha/\beta) < \infty$, there exists a partition $\gamma \in Z$ such that $\alpha = \beta\gamma$. Further, γ can be chosen so as to satisfy the inequalities*

$$H(\gamma) < H(\alpha/\beta) + 3\sqrt{H(\alpha/\beta)}.$$

PROOF. It is sufficient to consider the case $\alpha = \varepsilon$, because the general case can be reduced to this by factoring M by α. Since $H(\varepsilon/\beta) < \infty$, the conditional measures μ_B, $B \in M/\beta$, are discrete, and so there exist a measurable partition γ of M and a numbering $C_1, C_2, \ldots$ of its elements such that $\beta\gamma = \varepsilon$ and

$$\mu_B(C_1 \cap B) \geqslant \mu_B(C_2 \cap B) \geqslant \ldots \tag{44}$$

(see 1.10). We denote the numbers $\mu_B(C_n \cap B)$ by $m_n(B)$ and $\mu(C_n) = \int_{M/\beta} \mu_B(C_n \cap B)\, d\mu_B$ by m_n, respectively. (44) implies that $m_n(B) \leqslant 1/n$. Therefore $-\lg m_n(B) \geqslant \lg n$ and

$$H(\varepsilon/\beta) = -\int_{M/\beta} \sum_1^\infty m_n(B) \lg m_n(B)\, d\mu_B \geqslant \sum_1^\infty m_n \lg n. \tag{45}$$

We take any real number $s > 1$ and put $p_n = n^{-s}/\zeta(s)$, where $\zeta(s) = \sum_1^\infty n^{-s}$. Since $\sum_1^\infty p_n = 1$, for $\alpha_i = p_i$ and $x_i = m_i/p_i$, the inequality (10) in 4.7 holds (with $\varphi(x) = x \lg x$), that is, we have

$$\sum_1^\infty m_n \lg \frac{m_n}{p_n} \geqslant 0.$$

Consequently,

$$H(\gamma) = -\sum_1^\infty m_n \lg m_n \leqslant -\sum_1^\infty m_n \lg p_n =$$

$$= \sum_1^\infty m_n [\lg \zeta(s) + s \lg n] = \lg \zeta(s) + s \sum_1^\infty m_n \lg n \leqslant \lg \zeta(s) + sH(\varepsilon/\beta) \tag{46}$$

(see (45)) and so $H(\gamma) < \infty$.

This proves our first assertion. To prove the second we only need (46) and the rough estimate $\lg \zeta(s) < \frac{2}{s-1}$ which follows from the estimate $\zeta(s) < 1 + \int_1^\infty t^{-s}\,dt = 1 + (s-1)^{-1}$) and put $s = 1 + (H(\varepsilon/\beta))^{-\frac{1}{2}}$.

10.3. *For any two measurable partitions α and β such that $\alpha \geqslant \beta$ and $H(\alpha/\beta) \leqslant 1$, there exists a measurable partition γ such that $\alpha = \beta\gamma$ and $H(\gamma) < 4\sqrt{H(\alpha/\beta)}$.*

This follows from the preceding proposition if $H(\alpha/\beta) < 1$, then $H(\alpha/\beta) \leqslant \sqrt{H(\alpha/\beta)}$.

10.4. For a partition α of a space M and a set $B \subset M$, we denote by $\alpha \cap B$ the partition of M into sets $A \cap B$, where A is an element of α, and the set $M \setminus B$. Clearly, $\alpha \cap B \leqslant \alpha\beta$, where β is the partition with the elements B and $M \setminus B$. Thus, if $\alpha \in Z$ and B is measurable, then $\alpha \cap B \in Z$.

If $\alpha \in Z$, then for any positive δ there exists a positive λ such that $H(\alpha \cap B) < \delta$ for any set B with $\mu(B) < \lambda$.

PROOF. Let $A_1, A_2, \ldots$ be an enumeration of the elements of α and let m be sufficiently large so that

$$\sum_{k>m} [-\mu(A_k) \lg \mu(A_k)] < \delta_1 = \frac{\delta}{3} \tag{47}$$

and that $\mu(A_k) < e^{-1}$ for $k > m$. We take λ sufficiently small so that $-mt \lg t < \delta_1$ and $-\lg(1-t) < \delta_1$ for $0 < t < \lambda$ and we assume that $\mu(B) < \lambda$. Then the m-th partial sum of the series in

$$H(\alpha \cap B) = -\mu(M \setminus B) \lg \mu(M \setminus B) + \sum [-\mu(A_k \cap B) \lg \mu(A_k \cap B)],$$

is less than δ_1, and the same applies to the remainder of the sum, which is majorized by (47), and to the number $-\mu(M \setminus B) \lg \mu(M \setminus B)$. Consequently. $H(\alpha \cap B) < \delta$.

GENERATORS WITH FINITE ENTROPY

10.5. *If T is an aperiodic automorphism with finite entropy then, for any two partitions σ, $\xi \in Z$ and any positive δ there exists a partition $\eta \in Z$ such that $\eta_T \geqslant \xi_T$ and $H(\eta/\sigma^-) < h(T) - h(T, \sigma) + \delta$.*

PROOF. Let n be sufficiently large so that firstly, $H(\tau^n)/n - h(T, \tau) < \delta_1 = \delta/3$, where $\tau = \sigma\xi$, and secondly $-t \lg t - (1-t) \lg (1-t) < \delta_1$ if $0 < t < 1/n$. Let λ be sufficiently small so that $H(\xi \cap B) < \delta_1$ for any set B with $\mu(B) < \lambda$ (see **10.4**) and let C be a measurable set such that the sets

$$C,\ TC,\ \ldots,\ T^{n-1}C \tag{48}$$

are pairwise disjoint and the complement D of their union has measure less than λ (see **3.3**). If γ is the partition of M into the sets (48) and D, then

$$\sum_{k=0}^{n-1} H(\tau^n \cap T^k C/\sigma^n \cap T^k C) = \sum_{k=0}^{n-1} [H(\tau^n \cap T^k C) - H(\sigma^n \cap T^k C)] =$$
$$= H(\tau^n\gamma) - H(\sigma^n\gamma) + H(\sigma^n \cap D) - H(\tau^n \cap D).$$

Therefore one term of the sum on the left-hand side does not exceed the right-hand side divided by n and this term can be taken to be the term given by $k = 0$, because C can be replaced by any of the sets T^kC. Since

$$H(\tau^n\gamma) - H(\sigma^n\gamma) = H(\tau^n/\sigma^n\gamma) \leqslant H(\tau^n/\sigma^n) = H(\tau^n) - H(\sigma^n),$$

$$H(\sigma^n \cap D) - H(\tau^n \cap D) \leqslant 0,$$

and

$$\frac{1}{n} H(\tau^n) < h(T, \tau) + \delta_1 \leqslant h(T) + \delta_1, \quad \frac{1}{n} H(\sigma^n) \geqslant h(T, \sigma),$$

this gives the estimate $H(\tau^n \cap C/\sigma^n \cap C) < h(T) - h(T, \sigma) + \delta_1$. But $\sigma^n \cap C \leqslant \sigma^n\gamma_0$, where γ_0 is the partition of M into the sets C and $M\setminus C$, and since $\mu(C) < 1/n$, we have

$$H(\gamma_0) = -\mu(C) \lg \mu(C) - (1 - \mu(C)) \lg (1 - \mu(C)) < \delta_1.$$

Therefore

$$H(\xi^n \cap C/\sigma^-) \leqslant H((\tau^n \cap C)\gamma_0/\sigma^n) \leqslant H(\tau^n \cap C/\sigma^n\gamma_0) + H(\gamma_0) <$$
$$< H(\tau^n \cap C/\sigma^n \cap C) + \delta_1 < h(T) - h(T, \sigma) + 2\delta_1.$$

We put $\eta = (\xi^n \cap C)(\xi \cap D)$. Since $H(\xi \cap D) < \delta_1$, we have

$$H(\eta/\sigma^-) \leqslant H(\xi^n \cap C/\sigma^-) + H(\xi \cap D) < h(T) - h(T, \sigma) + \delta,$$

and all that remains is to verify that $\eta_T \geqslant \xi_T$. This follows from the relations

$$\xi \leqslant \xi\gamma = (\xi \cap D) \bigvee_{k=0}^{n-1} (\xi \cap T^kC) = (\xi \cap D) \bigvee_{k=0}^{n-1} T^k(T^{-k}\xi \cap C) \leqslant$$
$$\leqslant (\xi \cap D) \bigvee_{k=0}^{n-1} T^k(\xi^n \cap C) \leqslant \bigvee_{k=0}^{n-1} T^k\eta.$$

10.6. We denote by B_T the set of two-sided generators in Z for an automorphism T, and by Γ_T the set of partitions in Z for which $h(T, \xi) = h(T)$. Since the function $h(T, \xi)$ is continuous on Z, the set Γ_T is closed in Z and therefore is a complete metric space. By **9.4**, $B_T \subset \Gamma_T$.

B_T is a G_δ in Γ_T.

PROOF. We take a sequence $\alpha_1, \alpha_2, \ldots$ in Z such that $\alpha_n \nearrow \varepsilon$, and we denote by $B_T(p, q, r)$ the set of partitions $\xi \in \Gamma_T$ for which

$$H(\alpha_p / \bigvee_{k=-r}^{r} T^k\xi) < \frac{1}{q}.$$

This set is open in Γ_T, and

$$B_T = \bigcap_p \bigcap_q \bigcup_r B_T(p, q, r).$$

10.7. *If an automorphism T is aperiodic and has finite entropy, then it has a two-sided generator with finite entropy. Moreover, in this case $\Gamma_T \setminus B_T$ is a set of the first category in Γ_T.*

By **10.6** it is sufficient to show that if T is aperiodic and $h(T) < \infty$, then B_T is non-empty and dense in Γ_T.

I prove a little more: if $0 < \delta < 4$ and $\xi \in Z$ is a partition such that

$$h(T) - h(T, \xi) < \frac{\delta^2}{16},$$

then there exists a partition $\xi' \in B_T$ for which $\rho(\xi, \xi') < \delta$.

Let $\xi_0, \xi_1, \ldots$ be a sequence in Z such that

$$\xi_0 = \xi, \quad \xi_n \nearrow \varepsilon, \quad h(T) - h(T, \xi_h) < \frac{\delta^2}{2^{2k+4}}.$$

We construct partitions $\eta_1, \eta_2, \ldots$ such that

$$(\eta_h)_T \geqslant (\xi_k)_T, \quad H(\eta_k/(\xi_{k-1})_T) < \frac{\delta^2}{2^{2k+4}}$$

(see **10.5**) and then partitions $\zeta_1, \zeta_2, \ldots$ such that $(\xi_{k-1})_T\eta_k = (\xi_{k-1})_T\eta_k$ and $H(\zeta_k) < \delta/2^k$ (see **10.3**). Clearly, $(\xi_{k-1}\zeta_k)_T = (\xi_{k-1}\eta_k)_T \geqslant (\xi_k)_T$. Therefore

$$(\xi \bigvee_1^n \zeta_k)_T \geqslant (\xi_n)_T, \quad (\xi \bigvee_1^\infty \zeta_k)_T = \varepsilon,$$

and since

$$\rho(\xi, \xi \bigvee_1^\infty \zeta_h) = H(\bigvee_1^\infty \zeta_k) \leqslant \sum_1^\infty H(\zeta_k) < \delta,$$

we can put $\xi' = \xi \bigvee_1^\infty \zeta_k$.

10.8. *The entropy of an aperiodic automorphism is equal to the greatest lower bound of the entropies of the two-sided generators.*

This theorem requires proof only when $h(T) < \infty$, and then it is a consequence of Theorems 10.7 and 10.5; in the latter we have to take ξ to be a two-sided generator and σ to be the trivial partition ν.

10.9. *If the partition ζ is exhaustive under an aperiodic automorphism T with finite entropy, then T has a two-sided generator α with finite entropy satisfying $\alpha \leqslant \zeta$.*

PROOF. Let ξ be any two-sided generator with finite entropy, let $\xi_1, \xi_2, \ldots$ be a sequence in Z, and $n_1, n_2, \ldots$ an increasing sequence of positive integers such that

$$\xi_k \leqslant T^{n_k}\zeta, \quad \rho(\xi, \xi_k) < \frac{1}{2^{2k+5}}.$$

From the last inequality it follows that:

$$H(\xi_{k+1}/\xi_k) \leqslant \rho(\xi_k, \xi_{k+1}) < \frac{1}{2^{2k+4}},$$

and so there exist partitions $\eta_1, \eta_2, \ldots$ such that $\xi_k\xi_{k+1} = \xi_k\eta_k$ and $H(\eta_k) < 2^{-k}$ (see **10.3**). We put

$$\alpha = T^{-n_1}\xi_1 \bigvee_{k=1}^\infty T^{-n_{k+1}}\eta_k.$$

Since $\xi_1 \leqslant T^{n_1}\zeta$ and $\eta_k \leqslant \xi_k\xi_{k+1} \leqslant T^{n_{k+1}}\zeta$, we have $\alpha \leqslant \zeta$. Since $H(\alpha) \leqslant H(\xi_1) + \sum_1 H(\eta_k) < H(\xi_1) + 1$, we have $\alpha \in Z$. Finally, $\xi_k\eta_k \geqslant \xi_{k+1}$ implies that

$$\xi_1 \bigvee_1^\infty \eta_k \geqslant \bigvee_1^\infty \xi_k, \quad \alpha_T = (\xi_1 \bigvee_1^\infty \eta_k)_T \geqslant (\bigvee_1^\infty \xi_k)_T,$$

and since for any n

$$H(\xi/\bigvee_1^\infty \xi_k) \leqslant H(\xi/\xi_n) \leqslant \rho(\xi, \xi_n),$$

we have

$$H(\xi/\bigvee_1^\infty \xi_k) = 0, \quad \bigvee_1^\infty \xi_k \geqslant \xi, \quad \alpha_T \geqslant \xi_T = \varepsilon.$$

10.10. *Let T be an automorphism with finite entropy and ζ be an invariant partition with $h(T, \zeta) = h(T)$. If ξ is a two-sided generator with finite entropy such that $\xi T^{-1}\zeta = \zeta$, then $\xi^- = \zeta$.*

By **6.3** and **6.6** it is sufficient to prove that

$$H(\eta/\xi^-) = H(\eta/\zeta) \tag{49}$$

for any measurable partition η satisfying $\eta \leqslant T^n\xi^-$ for some n. But by **8.4**, $H(T^n\zeta/\zeta) = nh(T, \zeta) = nh(T)$, and by **8.4** and **9.4**, $H(T^n\xi^-/\xi^-) = nh(T, \xi) = nh(T)$. Moreover, from $\xi T^{-1}\zeta = \zeta$ it follows (by induction) that $T^n\zeta = \zeta T^n\xi^-$, and hence $H(T^n\zeta/\zeta) = H(T^n\xi^-/\zeta)$. This proves (49) for $\eta = T^n\xi^-$. If $\eta < T^n\xi^-$, then

$$H(\eta/\xi^-) = H(T^n\xi^-/\xi^-) - H(T^n\xi^-/\eta\xi^-) \leqslant H(T^n\xi^-/\zeta) - H(T^n\xi^-/\eta\zeta) = H(\eta/\zeta),$$

the converse inequality is obvious.

10.11. *If an endomorphism T is aperioduc and $h(T, \varepsilon) = h(T) < \infty$, then T has a generator with finite entropy. More generally, if the factor-endomorphism T_ζ of an endomorphism T is aperiodic and $h(T, \zeta) = h(T_\zeta) < \infty$, then there exists a partition $\xi \in Z$ such that $\xi^- = \zeta$.*

PROOF. We may restrict ourselves to the case when T is an automorphism and ζ is an exhaustive partition. For, the endomorphism T can be replaced by the natural extension of the factor-endomorphism T_ζ (see **3.7**).

Let α be a two-sided generator in Z satisfying $\alpha \leqslant \zeta$ (see **10.9**) and β be a partition in Z such that $\beta T^{-1}\zeta = \zeta$ (see **10.2**). We put $\xi = \alpha\beta$. This is a two-sided generator with finite entropy, and clearly $\xi T^{-1}\zeta = \zeta$. By **10.10**, $\xi^- = \zeta$.

COUNTABLE GENERATORS

10.12. We denote by $m(\xi)$ the greatest measure of the elements of the measurable partition ξ.

Let T be an aperiodic endomorphism and β a measurable partition such that $\beta T^{-1}\varepsilon = \varepsilon$. If β is finite or countable, then for finite measurable partition α and any positive δ there exists a finite measurable partition γ such that $\beta^-\gamma^- \geqslant \alpha$ and $m(\gamma) > 1 - \delta$.

PROOF. By **3.3** there exists a set A and a positive integer n such that

$$\bigcup_0^{n-1} T^k A = M, \qquad \mu(A) < \frac{\delta}{2}.$$

Since β is finite or countable, β^s is at most countable, and so there exists a set B such that $\mu(B) > 1 - \frac{\delta}{2}$, and in which β^n induces a finite partition

$(\beta^n)_B$. Let $B_1^s, \ldots, B_{r_s}^s$ be the elements of the partition induced by β^s, with $s \leqslant n$, in the set $B \cap T^{-(s-1)}A$, and let α_i^s be the partition induced in B_i^s by α. The endomorphism T^{s-1} is a one-to-one transformation on the B_i^s in A and hence carries α_i^s to a well-defined partition of some part of A. To this partition we add, as new elements, the remaining part of A and the set $M \setminus A$, denoting the resulting finite measurable partition of M by γ_i^s, and we put

$$\gamma = \left(\bigvee_{s=1}^{n} \bigvee_{i=1}^{r_s} \gamma_i^s\right) (\alpha \cap (M \setminus B)).$$

Clearly, $\beta^- \gamma^- \geqslant \alpha$ and $m(\gamma) = \mu(B \setminus A) > 1 - \delta$.

10.13. *If an endomorphism T is aperiodic and the inverse images of points are at most countable* (mod 0), *then T has a finite or countable generator. In particular, every aperiodic automorphism has a finite or countable generator.*

PROOF. Let β be a finite or countable measurable partition such that $\beta T^{-1}\varepsilon = \varepsilon$ (see **1.10**), $\alpha_1, \alpha_2, \ldots$ finite measurable partitions with product ε, and $\gamma_1, \gamma_2, \ldots$ finite measurable partitions such that $\beta^- \gamma_n^- \geqslant \alpha_n$ and $m(\gamma_n) > 1 - 2^n$ (see **10.12**). We put $\gamma = \bigvee_1^\infty \gamma_k$, $\xi = \beta\gamma$. If C_n is an element of γ_n with measure $m(\gamma_n)$, then each of the sets

$$C_n' = \bigcap_{k=n}^{\infty} C_k$$

is the sum of finitely many elements of γ, and since

$$\mu(C_n') > 1 - \sum_{k=n}^{\infty} 2^{-k} = 1 - 2^{-(n-1)},$$

the sets C_n' cover M (mod 0). Therefore γ is finite or countable, and so ξ is finite or countable. Finally,

$$\xi^- = \bigvee_{k=1}^{\infty} \beta^- \gamma_k^- \geqslant \bigvee_1^{\infty} \alpha_k = \varepsilon.$$

§11. Automorphisms with zero entropy

11.1. *An endomorphism has zero entropy if and only if every invariant partition is completely invariant, that is, every factor-endomorphism is an automorphism. In particular, every endomorphism with zero entropy is an automorphism.*

PROOF. If a partition ζ is invariant under an endomorphism T of zero entropy, then

$$H(\zeta / T^{-1}\zeta) = h(T, \zeta) \leqslant h(T) = 0,$$

and so $T^{-1}\zeta = \zeta$. Conversely, if every invariant partition is completely invariant, then for any invariant partition ζ

$$h(T, \zeta) = H(\zeta / T^{-1}\zeta) = 0,$$

and therefore $h(T) = 0$.

11.2. *The following statements are equivalent for automorphisms*:

A. *The entropy is zero.*

B. *The only exhaustive partition is* ε.

C. *Every two-sided generator is a generator.*

The equivalence of B and C is obvious. The implication $A \to B$ is contained in Proposition 11.1. The converse implication $B \to A$ requires additional techniques and will be established in **12.6**.

11.3. *An automorphism has zero entropy if and only if it has a generator with finite entropy.*

PROOF. If T is an automorphism with zero entropy, then by **10.7**, T has a two-sided generator with finite entropy, and by **11.2** it is a generator.

Conversely, if T is an automorphism with a generator $\xi \in Z$, then

$$h(T) = h(T, \xi) = H(\xi/T^{-1}\xi^{-}) = H(\xi/T^{-1}\varepsilon) = H(\xi/\varepsilon) = 0.$$

11.4. *The generators of an automorphism with zero entropy form an everywhere dense* G_δ *in* Z.

This follows from **10.6** and **10.7**, because for an automorphism with zero entropy $\Gamma_T = Z$.

11.5. *Among all the factor-automorphisms with zero entropy of an arbitrary endomorphism* T *there is a maximal one* T_π. *In other words, every endomorphism* T *has a completely invariant partition* $\pi = \pi(T)$ *such that* $h(T_\pi) = 0$ *and if* $H(T_\zeta) = 0$, *then* $\zeta \leqslant \pi$.

PROOF. We take π to be the product of all completely invariant partitions ζ for which $h(T_\zeta) = 0$ and show that $h(T_\pi) = 0$. Let Π be the set of partitions $\xi \in Z$ with $h(T, \xi) = 0$ and $\xi_1, \xi_2, \dots$ be a dense sequence in Π. We put $\eta_n = \bigvee_1^n \xi_k$, and $\eta = \bigvee_1^\infty \xi_k$. If $\xi \in \Pi$, then $H(\xi/\eta) \leqslant H(\xi/\xi_n) \leqslant \rho(\xi, \xi_n)$ for any n, and since $\rho(\xi, \xi_n)$ takes arbitrarily small values, $H(\xi/\eta) = 0$. Thus, if $\xi \in \Pi$, then $\xi \leqslant \eta$, and so η coincides with the product of all partitions in Π. But, clearly, this product is π. Consequently, $\eta_n \nearrow \pi$ and by Theorem 9.5 the equation $h(T_\pi) = 0$ will be proved if we show that $h(T, \eta_n) = 0$ for $n = 1, 2, \dots$ This follows from the equations $h(T, \xi_k) = 0$ and Theorem 8.2.

11.6. The simplest examples of automorphisms with zero entropy – the identity automorphism and periodic automorphisms – were mentioned in **9.10**. As we shall see in **14.4**, all automorphisms with discrete or singular spectrum and all automorphisms with a spectrum of finite multiplicity have zero entropy. There is another curious class of automorphisms with zero entropy, which is not discussed in these lectures. This is the class of automorphisms with quasi-discrete spectrum studied by L.M. Abramov [4].

For Lebesgue spaces with continuous measure, automorphisms with zero entropy form an everywhere dense G_δ in the space of automorphisms. For details see [23].

§12. The theory of invariant partitions

12.1. *If* T *is an automorphism and* ζ *is an exhaustive partition, then* $\bigwedge_0^\infty T^{-n}\zeta \geqslant \pi(T)$ (see **11.5**).

PROOF. We put $\bigwedge_1^\infty T^n\zeta = \zeta_0$. It is sufficient to prove that if $\eta \leqslant \pi(T)$ and $\eta \in Z$, then $\eta \leqslant \zeta_0$. For this it is sufficient to prove that

$$H(\xi/\zeta_0\eta_T) = H(\xi/\zeta_0) \tag{50}$$

for every partition $\xi \in Z$ that is coarser than one of the partitions $T^m\zeta$; for since $T^n\zeta \nearrow \varepsilon$, the set of partitions coarser than the partitions $T^m\zeta$, is dense in Z (see **6.3**), and hence (50) implies that $\xi_0\eta_T = \zeta_0$ (see **6.6**).

We prove (50). For every positive integer p

$$H(\xi/\zeta_0) \geqslant H(\xi/\zeta_0\eta_T) \geqslant H(\xi/T^{-p}\xi_{T^p}^-\zeta_0\eta_T) \tag{51}$$

(where $\xi_{T^p}^- = \bigvee_{k=0}^\infty T^{-pk}\xi$; see **3.4**). As the partitions ζ_0 and η_T^- are completely invariant, for every positive integer n we have

$$T^{-p}\xi_{T^p}^-\zeta_0 = T^{-p}(\xi\zeta_0)_{T^p}^-, \quad \eta_T = T^{-pn}(\eta_T^p)_{T^p}^-$$

and

$$H(\xi/T^{-p}\xi_{T^p}^-\zeta_0\eta_T) = H(\xi/T^{-p}(\xi\zeta_0)_{T^p}^-\cdot T^{-pn}(\eta_T^p)_{T^p}^-).$$

This enables us to apply Theorem 7.6, with T^p in place of T, to the right-hand side of (51). The result is:

$$H(\xi/T^{-p}\xi_{T^p}^-\cdot\zeta_0\cdot\eta_T) = H(\xi/T^{-p}\xi_{T^p}^-\cdot\zeta_0).$$

Since $\xi \leqslant T^m\zeta$, we have $T^{-p}\xi_{T^p}^-\zeta_0 \searrow \zeta_0$. Therefore $H(\xi/T^{-p}\xi_{T^p}^-\zeta_0) \nearrow H(\xi/\zeta_0)$, and (50) follows from (51).

12.2. *If the partition ζ is invariant under an endomorphism T and $\eta = \bigwedge_0^\infty T^{-k}\zeta$, then*

$$h(T, \zeta) + h(T_\eta) \leqslant h(T_\zeta). \tag{52}$$

PROOF. Since every endomorphism has a natural extension (see **3.7**) we need only consider the case when T is an automorphism. Let $\zeta_1, \zeta_2, \ldots$ be partitions in Z such that $\zeta_n \nearrow \zeta$ and ξ a partition such that $\xi \leqslant \eta$. By **7.7**,

$$H(\zeta_n/T^{-1}\zeta_n^-\cdot\xi_T) + h(T, \xi) = h(T, \zeta_n\xi), \tag{53}$$

and since $\zeta_n^- \leqslant \zeta^- = \zeta$, $\xi_T \leqslant \eta_T = \eta \leqslant T^{-1}\zeta$, we see that $T^{-1}\zeta_n^-\xi_T \leqslant T^{-1}\zeta$, $H(\zeta_n/T^{-1}\zeta_n^-\xi_T) \geqslant H(\zeta_n/T^{-1}\zeta)$. The right-hand side of the last inequality tends to $h(T, \zeta)$ as $n \to \infty$ and the right-hand side of (53) does not exceed $h(T_\zeta)$. Consequently,

$$h(T, \zeta) + h(T, \xi) \leqslant h(T_\zeta),$$

from which we obtain (52).

12.3. *If the partition ζ is invariant under an endomorphism T and $h(T, \zeta) = h(T_\zeta) < \infty$, then $\bigwedge_0^\infty T^{-n}\zeta \leqslant \pi(T)$. In particular, for any partition $\xi \in Z$,*

$$\bigwedge_0^\infty T^{-n}\xi^- \leqslant \pi(T).$$

This is a corrollary of the preceding theorem.

12.4. A partition ζ is said to be *extremal* with respect to an automorphism T if it is exhaustive and $T^{-n}\zeta \searrow \pi(T)$. In other words, ζ is extremal if

$$T\zeta \geqslant \zeta, \quad \bigvee_0^\infty T^n\zeta = \varepsilon, \quad \bigwedge_0^\infty T^{-n}\zeta = \pi(T).$$

Theorems 12.1 and 12.3 show that *if a partition ζ is exhaustive for an automorphism T and $h(T, \zeta) = h(T) < \infty$, then ζ is extremal.* In particular, if ξ is a two-sided generator with finite entropy, then ξ^- is an extremal partition. (see **9.4.**)

12.5. *Every automorphism T has an extremal partition ζ such that $h(T, \zeta) = h(T)$.*

If $h(T) < \infty$, then we can put $\zeta = \xi^-$, where ξ is a two-sided generator in Z. The following proof (which was found before Theorem 10.7, see [24]) works for $h(T) < \infty$ as well as for $h(T) = \infty$.

Let $\xi_1, \xi_2, \ldots$ be a sequence in Z such that $\xi_n \nearrow \varepsilon$, and $n_1, n_2, \ldots$ a sequence of integers. We put

$$\eta_p = \bigvee_{k=1}^p T^{-n_k}\xi_k, \quad \eta = \bigvee_{k=1}^\infty T^{-n_k}\xi_k, \quad \zeta = \eta^-.$$

Clearly, ζ is an exhaustive partition. We show that if $n_1, n_2, \ldots$ increases sufficiently rapidly, then

$$H(\eta_p/T^{-1}\eta_p^-) - H(\eta_p/T^{-1}\zeta) \longrightarrow 0, \tag{54}$$

and that if this condition is satisfied, then ζ is extremal with $h(T, \zeta) = h(T)$.

The relation (54) holds necessarily if we choose $n_1, n_2, \ldots$ subject to the following condition:

$$H(\eta_p/T^{-1}\eta_{q-1}^-) - H(\eta_p/T^{-1}\eta_q^-) < \frac{1}{p}\,\frac{1}{2^{q-p}} \quad \text{for } p < q. \tag{55}$$

For then

$$H(\eta_p/T^{-1}\eta_p^-) - H(\eta_p/T^{-1}\eta_q^-) = \sum_{k=0}^{q-p-1} [H(\eta_p/T^{-1}\eta_{p+k}^-) - H(\eta_p/T^{-1}\eta_{p+k+1}^-)] <$$

$$< \frac{1}{p}\sum_{k=0}^{q-p-1} \frac{1}{2^{k+1}} < \frac{1}{p},$$

and since $H(\eta_p/T^{-1}\eta_q^-) \longrightarrow H(\eta_p/T^{-1}\zeta)$ as $q \to \infty$ (see **5.11**),

$$H(\eta_p/T^{-1}\eta_p^-) - H(\eta_p/T^{-1}\zeta) \leqslant \frac{1}{p}.$$

The choice can be made by induction: if $n_1, \ldots, n_{q-1}$ have been chosen, n_q can be chosen sufficiently large so that the inequalities (55) are satisfied with $p = 1, \ldots, q-1$. This is possible by **7.6**.

Since $(\eta_p)_T = (\xi_p)_T$ and $\xi_p \nearrow \varepsilon$, we have $H(\eta_p/T^{-1}\eta_p^-) = h(T, \eta_p) = h(T, \xi)_p \longrightarrow h(T)$. Since $\eta_p \nearrow \eta$, we have $H(\eta_p/T^{-1}\zeta) \longrightarrow H(\eta/T^{-1}\zeta) = H(\zeta/T^{-1}\zeta)$. Therefore (54) implies that $h(T, \zeta) = h(T)$.

Since ζ is an exhaustive partition, by **12.1** $\bigwedge_0^\infty T^{-n}\zeta \geqslant \pi(T)$, and it remains to show that (54) implies the converse inequality. If $h(T) < \infty$, then by **12.3** the converse inequality follows from the fact that $h(T, \zeta) = h(T)$. The following argument also works for $h(T) = \infty$. It is sufficient to establish that any partition $\alpha \in Z$ satisfying the inequality $\alpha \leqslant \bigwedge_0^\infty T^{-n}\zeta$, also satisfies the inequality $\alpha \leqslant \pi(T)$. Expanding $h(T, \eta_p\alpha)$ in two ways by formula (28) we find

$$h(T, \alpha) = H(\alpha/T^{-1}\alpha^-(\eta_p)_T) + H(\eta_p/T^{-1}\eta_p^-) - H(\eta_p/T^{-1}\eta_p^-\alpha_T).$$

Since $(\eta_p)_T \nearrow \varepsilon$, we have $H(\alpha/T^{-1}\alpha^-(\eta_p)_T) \to 0$, and the difference $H(\eta_p/T^{-1}\eta_p^-) - H(\eta_p/T^{-1}\eta_p^-\alpha_T)$ does not exceed the difference in (54), because $T^{-1}\eta_p^-\alpha_T \leqslant T^{-1}\zeta$, and so this also tends to zero. Consequently, $h(T, \alpha) = 0$ and $\alpha \leqslant \pi(T)$.

12.6. *If an automorphism T does not have an exhaustive partition other than ε, then $h(T) = 0$.*

PROOF. If ζ is an extremal partition for T (see **12.5**), then on the one hand $\zeta = \varepsilon$, and on the other hand $\bigwedge_0^\infty T^{-n}\zeta = \pi(T)$. Consequently $\pi(T) = \varepsilon$ and $h(T) = 0$.

12.7. Examples show that there exist extremal partitions ζ such that $h(T, \zeta) < h(T)$ (see [13], [36]). It is not excluded that for any automorphism T and any number c with $0 < c \leqslant h(T)$ there exists an extremal partition ζ for which $h(T, \zeta) = c$. However, this generalization of Theorem 12.5 is not proved even for a single automorphism with positive entropy.

§13. Endomorphisms with completely positive entropy

13.1. We say that an endomorphism T has *completely positive entropy* if each of its factor-endomorphisms T_ζ with $\zeta \neq \nu$ has positive entropy. An equivalent condition: $\pi(T) = \nu$. Another equivalent condition: $h(T, \xi) > 0$ if $\xi \in Z$ and $\xi \neq \nu$.

An endomorphism with completely positive entropy is ergodic. For if an endomorphism T is not ergodic, then it has a fixed partition $\xi \neq \nu$, and the corresponding factor-endomorphism T_ξ has zero entropy, because it is the identity.

13.2. *If the factor-endomorphism T_ζ of an endomorphism T has completely positive entropy, then ζ and $\pi(T)$ are independent.*

It is sufficient to show that, for any two partitions $\xi, \eta \in Z$ such that $\xi \leqslant \zeta$, $\eta \leqslant \pi$, the following holds:

$$H(\xi/\eta) = H(\xi) \tag{56}$$

(see **5.10**).

For any positive integer p,

$$H(\xi) \geqslant H(\xi/\eta) \geqslant H(\xi/T^{-p}\xi_T^-\mathfrak{p}\eta^-). \tag{57}$$

As $\eta \leqslant \pi$, the partition η^- is completely invariant (see 11.1) and so, for any positive integer n,

$$\eta^- = T^{-pn}(\eta_T^p)_{T^p}^-, \quad H(\xi/T^{-p}\xi_{T^p}^-\eta^-) = H(\xi/T^{-p}\xi_{T^p}^- T^{-pn}(\eta_T^p)_{T^p}^-).$$

This makes it possible to apply Theorem 7.6, with T^p in place of T, to the right-hand side of (57). The result: $H(\xi/T^{-p}\xi_{T^p}^-\eta^-) = H(\xi/T^{-p}\xi_{T^p}^-)$. As $\pi(T_\zeta)$ is trivial and $\xi \leqslant \zeta$, we have $T^{-p}\xi^- \searrow \nu$ (see 12.3). Therefore $H(\xi/T^{-p}\xi_{T^p}^-) \geqslant H(\xi/T^{-p}\xi^-) \nearrow H(\xi)$, and (56) follows from (57).

13.3. *If the set of partitions $\xi \in Z$ for which the factor-endomorphism T_{ξ^-} has completely positive entropy is dense in Z, then the endomorphism T has completely positive entropy.*

PROOF. If the factor-endomorphism T_{ξ^-} has completely positive entropy, then $H(\xi/\pi(T)) = H(\xi)$ by Theorems 13.2 and 5.10. Consequently, the set of partitions ξ for which this equation is true is dense in Z and by Theorem 6.6, $\pi(T) = \nu$.

13.4. *If the factor-endomorphisms $T_{\zeta_1}, T_{\zeta_2}, \ldots$ of an endomorphism T have completely positive entropy and $T_{\zeta_n} \nearrow T$, then T has completely positive entropy.*

This follows from Theorems 13.3 and 6.3.

13.5. *Every endomorphism T has a maximal factor-endomorphism with completely positive entropy. (The factor-endomorphism is a factor-automorphism if T is an automorphism.)*

This follows from the preceding theorem and Zorn's Lemma.

13.6. M.S. Pinsker, who discovered Theorem 13.2, suggests that it might be possible to decompose every ergodic automorphism into the direct product of an automorphism with completely positive entropy and an automorphism with zero entropy (see [17]). Whether this is true is not known so far. Examples show that a maximal factor-automorphism with completely positive entropy (which exists for every automorphism by 13.5) is not unique.

13.7. *If the largest factor-automorphism T_α of an endomorphism T* (see 3.5) *has completely positive entropy, then T also has completely positive entropy.*

PROOF. As T_π is a factor-automorphism and T_α is the largest factor-automorphism of T, we have $\pi(T) \leqslant \alpha(T)$. Consequently, if T_α is an automorphism with completely positive entropy, then $\pi(T) = \nu$.

13.8. *The natural extension of an endomorphism with completely positive entropy is an automorphism with completely positive entropy.*

This follows from Theorem 13.4: if ζ is an exhaustive partition under an automorphism T, then $T_{T^n\zeta} \nearrow T$, and all these factor-endomorphisms are isomorphic to T_ζ.

Another proof depends on Theorem 12.1: if ζ is an exhaustive partition under an automorphism T, then

$$\pi(T) \leqslant \bigwedge_0^\infty T^{-n}\zeta \leqslant \zeta,$$

and so the fact that the factor-endomorphism T_ζ has completely positive entropy implies that $\pi(T) = \nu$.

EXACT ENDOMORPHISMS AND K-AUTOMORPHISMS

13.9. An endomorphism T is said to be *exact* if $\alpha(T) = \nu$, that is, if T has no non-trivial factor-automorphisms. An equivalent condition:

$$\bigcap_0^\infty T^{-n}\mathfrak{M} = \mathfrak{N}. \tag{58}$$

13.7 implies that *exact endomorphisms have completely positive entropy*. In particular, they are ergodic. This can be seen directly from (58): if a set is measurable and invariant under T, then it is contained in the algebra (58) and so has measure 0 or 1.

Bernoulli endomorphisms are examples of exact endomorphisms.

13.10. If T is an automorphism with completely positive entropy, then by Theorem 12.5 there exists a partition ζ with the properties:

$$T\zeta \geqslant \zeta, \quad \bigvee_0^\infty T^n\zeta = \varepsilon, \quad \bigwedge_0^\infty T^{-n}\zeta = \nu, \tag{59}$$

By 12.1, every automorphism T having a partition ζ with these properties has completely positive entropy.

The conditions (59) are older than the theory of invariant partitions given in §12: they occur in Kolmogorov's work [13], starting from the entropy theory of measure-preserving transformations. Starting from probability theory arguments Kolmogorov called automorphisms having a partition with the properties (59) *quasi-regular*. Later they were called *Kolmogorov automorphisms or K-automorphisms*. Thus, we may say that *the class of automorphisms with completely positive entropy coincides with that of K-automorphisms*.

As examples of K-automorphisms we can take Bernoulli automorphisms: if ξ is the generator of a Bernoulli automorphism given in **3.6**, then the partition $\zeta = \xi^-$ satisfies (59).

13.11. It is clear that the direct product of K-automorphisms is a K-automorphism. As the natural extension of any endomorphism with completely positive entropy is a K-automorphism (see **13.8** and **13.10**), the direct product of endomorphisms with completely positive entropy has completely positive entropy.

§14. Entropy and the spectrum

14.1. LEMMA. *If for any measurable partition ζ of a space M there exists a set A of positive measure without points of positive measure and with $\zeta_A \neq \varepsilon_A$, then the subspace $L_2(M) \ominus L_2(M, \zeta)$ is infinite-dimensional.*

We preceed the proof with the following obvious remark: if X is an arbitrary set of positive measure in M, then the subspace of $L_2(M)$ consisting of functions equal to zero outside X can be identified with the canonically isomorphic space $L_2(X)$ of all square-integrable functions on the subspace X of M (see **1.2**).and we have

$$L_2(X) \ominus L_2(X, \zeta_X) \subset L_2(M) \ominus L_2(M, \zeta). \tag{60}$$

First we prove the lemma under the assumption that $A = M$. Since $\zeta \neq \varepsilon$, we have $L_2(M, \zeta) \neq L_2(M)$ and in $L_2(M)$ there exists a function $\varphi \neq 0$ orthogonal to $L_2(M, \zeta)$. Let B be the set of points $x \in M$ for which $\varphi(x) \neq 0$. By (60) (with $X = B$) it is sufficient to show that $\dim(L_2(B) \ominus L_2(B, \zeta_B)) = \infty$. If $\dim L_2(B, \zeta_B) < \infty$, then this is implied by $\dim L_2(B) = \infty$. If $\dim L_2(B, \zeta_B) = \infty$, then $L_2(B, \zeta_B)$ contains an infinite sequence of linearly independent orthogonal functions $g_1, g_2, \ldots$ and $\dim(L_2(B) \ominus L_2(B, \zeta_B)) = \infty$ because $L_2(B) \ominus L_2(B, \zeta_B)$ contains the infinite sequence of linearly independent functions $g_1\varphi, g_2\varphi, \ldots$

The case $A \neq M$ reduces to $A = M$ if by using (60) with $X = A$ we go over from M with partition ζ to A with partition ζ_A.

14.2. *If T is an endomorphism of a space M and ζ is a measurable partition finer than $\pi(T)$ other than ε, then the space $L_2(M) \ominus L_2(M, \zeta)$ is infinite-dimensional.*

By the previous theorem it is sufficient to prove that there exists a set $A \subset M$ of positive measure and without points of positive measure such that $\zeta_A \neq \varepsilon_A$. If this is not the case, then there exists an element of ζ and hence an element of $\pi(T)$ containing two points of positive measure. Let γ be the partition of M into one of these points and its complement. Clearly, γ cannot be coarser than $\pi(T)$, but the factor-endomorphism T_γ- is a periodic automorphism and so has zero entropy.

14.3. *If T is an automorphism of a space M, then the operator U_T has a Lebesgue spectrum of infinite multiplicity in the subspace $L_2(M) \ominus L_2(M, \pi)$*

PROOF. Let ζ be an extremal partition for T. For any integer n we put

$$H_n = U_T^n L_2(M, \zeta) \ominus U_T^{n+1} L_2(M, \zeta).$$

Since

$$U_T^n L_2(M, \zeta) = L_2(M, T^{-n}\zeta) \begin{cases} \nearrow L_2(M) & \text{as } n \longrightarrow \infty, \\ \searrow L_2(M, \pi) & \text{as } n \longrightarrow -\infty, \end{cases}$$

we have

$$L_2(M) \ominus L_2(M, \pi) = \bigoplus_{n=-\infty}^{+\infty} H_n. \tag{61}$$

By Proposition 14.2 applied to the factor-endomorphism T_ζ the subspace H_0 is infinite-dimensional. Let $f_1, f_2, \ldots$ be a basis in H_0. (61) shows that the functions $U^n f_k$ form a basis in $L_2(M) \ominus L_2(M, \pi)$. Thus, U_T has a Lebesgue spectrum of infinite-multiplicity in $L_2(M) \ominus L_2(M, \pi)$.

14.4. COROLLARIES. *An automorphism with completely positive entropy has a Lebesgue spectrum of infinite multiplicity.*

Automorphisms with discrete spectrum have zero entropy.

Automorphisms with singular spectrum have zero entropy.

Automorphisms with a spectrum of finite-multiplicity have zero entropy.

14.5. There exist automorphisms with zero entropy having a Lebesgue spectrum of infinite multiplicity. The first example of this kind was constructed in 1959 by Girsanov, but was not published. There is also an example recently published by Newton and Parry [49].

One of the classical unsolved problems of the theory of measure-preserving transformations is: what spectral properties must a unitary operator U_T

have to be adjoint to an ergodic automorphism? Theorem 14.3 reduces this to the problem of ergodic automorphisms with zero entropy.

14.6. We turn now to *endomorphisms*. According to **1.6** and **2.6** the equation

$$\alpha(T) = \bigwedge_{n=0}^{\infty} T^{-n}\varepsilon$$

implies that

$$\bigcap_{n=0}^{\infty} U_T^n L_2(M) = L_2(M, \alpha),$$

and that the unitary part of U_T is canonically isomorphic to the operator U_{T_α} adjoint to T_α. **14.2** implies that *the defect of U_T is ∞ only if T is not an automorphism*. Combining these facts with Theorem 14.3 we see that the operator U_T adjoint to an arbitrary endomorphism T has the orthogonal decomposition:

$$U_T = U_{T_\alpha} \oplus W = U_{T_\pi} \oplus V \oplus W,$$

where V is a unitary operator with a Lebesgue spectrum of infinite-multiplicity and W is a semi-unitary operator with a homogenous spectrum of infinite multiplicity or zero. If T is an endomorphism with completely positive entropy, then U_{T_π} is the identity transformation on the one-dimensional space $C(M)$. If T is an automorphism with completely positive entropy or an exact endomorphism, then also $W = 0$ or $V = 0$. If T is an automorphism with zero entropy, then $V = W = 0$.

§15. Entropy and Mixing

15.1. LEMMA. *If T is an endomorphism, f_0 is a function in $L_2(M)$ orthogonal to $L_2(M, \pi(T))$ and $f_1, \ldots, f_r$ are bounded functions in $L_2(M)$, then for any sequence of complexes of non-negative integers*

$$(k_1^0, \ldots, k_1^r),\ (k_2^0, \ldots, k_2^r),\ \ldots, \tag{62}$$

satisfying the conditions

$$k_n^0 < k_n^1 < \ldots < k_n^r, \quad k_n^1 - k_n^0 \to \infty,$$

the following holds:

$$\left(\prod_{i=0}^{r} U_T^{k_n^i} f_i,\ 1\right) \to 0. \tag{63}$$

PROOF. We assume first that T is an automorphism and that the functions $f_1, \ldots, f_r$ are in $L_2(M, \zeta)$, where ζ is some extremal partition. Then the scalar product (63) has the form (f_0, g_n), where g_n is the complex conjugate of the product

$$\prod_{i=1}^{r} U^{k_n^i - k_n^0} f_i,$$

and since $k_n^i - k_n^0 < k_n^i - k_n^0$ for $i > 1$,

$$g_n \in U_T^{k_n^1 - k_n^0} L_2(M, \zeta) = L_2(M, T^{-(k_n^1 - k_n^0)}\zeta). \tag{64}$$

Therefore

$$|(f_0, g_n)| = |(P_n f_0, g_n)| \leqslant \| P_n f_0 \| \cdot \| g_n \|,$$

where P_n is the projection operator onto the subspace (64), and $T^{-n}\zeta \searrow \pi$ implies that

$$U_T^{k_n^1 - k_n^0} L_2(M, \zeta) \searrow L_2(M, \pi),$$

so that $\| P_n f_0 \| \to 0$. Since the functions $f_1, \ldots, f_r$ are bounded, the sequence of norms $\|g_n\|$ is bounded and so $(f_0, g_n) \to 0$.

In the general case Theorem 12.1 allows us to replace an endomorphism T by its natural extension, that is, to regard it as an automorphism, and Theorem 12.5 allows us to construct an extremal partition ζ_0 for this automorphism. We make a bounded approximation of $f_1, \ldots, f_n$ by functions in the subspace $U_T^{-p} L_2(M, \zeta_0) = L_2(M, T^p\zeta_0)$ for a sufficiently large p and apply the case already treated (with the same f_0 and with $\zeta = T^p\zeta_0$). So we get the complete result.

15.2. *Every endomorphism T is mixing on any sets $A_0, \ldots A_r$ independent of the partition $\pi(T)$.*

An equivalent formulation: *an endomorphism T is mixing on any bounded functions $f_0, \ldots, f_r$ independent of the partition $\pi(T)$.*

It is sufficient to show that for every sequence (62) satisfying the conditions

$$k_n^0 < k_n^1 < \ldots < k_n^r, \quad \min_{0 \leq i < j \leq r} (k_n^j - k_n^i) \to \infty,$$

the following holds:

$$\Big(\prod_{i=0}^{r} U_T^{k_n^i} f_i, 1\Big) \to \prod_{i=0}^{r} (f_i, 1). \tag{65}$$

The proof proceeds by induction on r. For $r = 0$ (65) is trivial. We assume that

$$\Big(\prod_{i=1}^{r} U_T^{k_n^i} f_i, 1\Big) \to \prod_{i=1}^{r} (f_i, 1). \tag{66}$$

The fact that f_0 is independent of $\pi(T)$ implies that its projection on $L_2(M, \pi)$ is equal to $(f_0, 1)$. Therefore the function $f_0' = f_0 - (f_0, 1)$ is orthogonal to $L_2(M, \pi)$ and by Proposition 15.1, the first term on the right-hand side of

$$\Big(\prod_{i=0}^{r} U_T^{k_n^i} f_i, 1\Big) = \Big(U_T^{k_n^0} f_0' \cdot \prod_{i=1}^{r} U_T^{k_n^i} f_i, 1\Big) + (f_0, 1)\Big(\prod_{i=1}^{r} U_T^{k_n^i} f_i, 1\Big)$$

converges to zero. Combining this with (66), we obtain (65).

COROLLARY. *An endomorphism with completely positive entropy is mixing of all orders.*

The converse of this theorem is not true; there exist automorphisms with zero entropy that are mixing of all orders. In particular, Girsanov's automorphism, mentioned in **14.5**, has this property.

§16. Entropy and the isomorphism problem

16.1. The first application of entropy in the theory of measure-preserving transformations was made by Kolmogorov in 1958 (see [13] and [14]). and was concerned with the partially solved *isomorphism problem*, that is, the problem of classifying automorphisms of a Lebesgue space with respect to isomorphism (mod 0).

Before Kolmogorov's work on this problem there had been no progress for quite some time. It has been known for some time that the classification of non-ergodic automorphisms reduces to the classification of the ergodic automorphisms, that two ergodic automorphisms with discrete spectra are isomorphic if and only if they are spectrally isomorphic and that this is not true for automorphisms with a mixed spectrum. This last fact was established with the help of special invariants organically connected with the presence in the spectrum of both discrete and continuous components. In the case of continuous spectrum the only available invariants were the spectral ones. It was not possible to establish that spectral isomorphism does not imply isomorphism.

This was proved by Kolmogorov with the help of entropy. As examples he used Bernoulli automorphisms: all these, apart from the trivial case, have Lebesgue spectra of infinite multiplicity and the entropy (or what comes to the same thing in this case, the invariant h_1; see **10.1**) can have any positive value.

16.2. At present automorphisms with completely positive entropy are the most interesting. This is due to their special position in both the general theory and in applications. In the first place, there is the isomorphism problem. We do not exclude the possibility that two automorphisms with completely positive entropy are isomorphic if their entropies are equal. If this were so, then every automorphism with completely positive entropy would be isomorphic to a Bernoulli automorphism.

The problem can be restricted to Bernoulli automorphisms. This case was first examined by Meshalkin [16] who proved that if the spaces of states of the Bernoulli automorphisms T and T' are finite and the probabilities of all the states are powers of a single rational number, then $h(T) = h(T')$ implies isomorphism. For example, spaces of states with probabilities 1/4, 1/4, 1/4, 1/4 and 1/2, 1/8, 1/8, 1/8, 1/8 produce isomorphic Bernoulli automorphisms. Meshalkin's work attracted the attention of many mathematicians; partial results have been obtained, but the complete problem is still unsolved.

16.3. Sinai also studied this problem (see [32], [34]). He proposed that one should consider a less stringent condition along with isomorphism: he calls two automorphisms *weakly isomorphic* if each is the homomorphic image of the other. Sinai's main result: if S is a Bernoulli automorphism with finite entropy not exceeding the entropy of an ergodic automorphism T, then S is a homomorphic image of T. In particular, two Bernoulli automorphisms with the same finite entropy are weakly isomorphic.

To appreciate the importance of weak isomorphism we note that two weakly isomorphic automorphisms have both the same entropy and the same spectral invariants. A systematic study of other invariants (contained in unpublished work by Yuzvinskii) shows that the situation is the same with these. Thus, at present we cannot distinguish between isomorphism and weak isomorphism. This applies to endomorphisms as well as to automorphisms. In measure theory there are simpler objects for which a complete classification exists, such as Lebesgue spaces, measurable partitions, and measurable functions, for which it is not difficult to show that weak isomorphism implies isomorphism.

16.4. The following problem appears to be more promising: does there exist a *universal* automorphism with completely positive entropy such that any automorphism with completely positive entropy is a homomorphic image? The natural candidate is the Bernoulli automorphism whose space of states has continuous measure. We do not exclude the possibility that this candidate is even isomorphic to its direct product with any automorphism with completely positive entropy.

16.5. Let us discuss the problem of isomorphism for endomorphisms. Here we have an obvious invariant, which is trivial in the case of an automorphism - the decreasing sequence of partitions

$$\varepsilon,\ T^{-1}\varepsilon,\quad T^{-2}\varepsilon,\ \ldots, \tag{67}$$

considered to within isomorphism. As was shown by Vinokurov, an endomorphism T is not specified by this invariant, taken to within equivalence, even in the class of exact endomorphisms with generators in Z. Whether much has to be added to the sequence (67) for a solution of the isomorphism problem, even for this class of endomorphisms, is not known. Vinokurov's proof does not contain explicitly defined new invariants.

There are entirely concrete situations in which this problem is of interest. Suppose, for example, that T is a group endomorphism of the two-dimensional torus with eigenvalues λ_1 and λ_2 and that S is a Bernoulli endomorphism with space of states consisting of $|\lambda_1\lambda_2|$ points of measure $|\lambda_1\lambda_2|^{-1}$. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then the endomorphism T is exact and has a finite generator, and the sequence (67) is isomorphic to the sequence ε, $S^{-1}\varepsilon$, $S^{-2}\varepsilon$, ... Are S and T isomorphic?

The preceding arguments call for a more intensive study of decreasing sequences of measurable partitions. At present there exists a complete metric classification of such sequences. The classification of certain measurable partitions has been known for twenty years (see [20]); finite decreasing sequences were discussed by Guseva [12]; the difficult transition to infinite sequences was recently carried out by Vershik. Vershik's main result: sequences $\{\xi_k\}_1^\infty$ and $\{\eta_k\}_1^\infty$ are isomorphic if $\xi_n \searrow \nu$ and $\eta_n \searrow \nu$ and the sequences $\{\xi_k\}_1^n$ and $\{\eta_k\}_1^n$ are isomorphic for any n.

It would be interesting to clarify what conditions a sequence $\xi_1, \xi_2, \ldots$ must fulfill for the existence of an exact endomorphism T with $T^{-n}\varepsilon = \xi_n$ $(n = 1, 2, \ldots)$. The first thing to do is to prove or disprove the statement: in a space with continuous measure for any measurable partition ξ having no elements of positive measure there exists an exact endomorphism T with $T^{-1}\varepsilon = \xi$.

If an endomorphism is not exact, the question arises whether it can be decomposed into the direct product of an automorphism and an exact endomorphism. This problem is related to Pinsker's problem (see **13.6**).

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APPENDIX

METRIC PROPERTIES OF ENDOMORPHISMS OF LOCALLY-COMPACT GROUPS[1]

S.A. YUZVINSKII

Let G be a locally-compact group, μ a Haar measure on G, and T an endomorphism of G. Then T is a measurable transformation which can be studied from the point of view of measure theory.

First we make stronger assumptions: the group G is compact and has a countable topological basis, and T maps G onto G. In this case G is a Lebesgue space and T, because of uniqueness of the Haar measure, is an endomorphism in the measure-theoretical sense.

To G there corresponds a set X of equivalence classes of irreducible representations, and T indices the transformation U: $X \to X$ defined for a representation A by the formula $UA(g) = A(Tg)$, $g \in G$, and clearly this can be carried over to the equivalence classes. If G is commutative, then X becomes its character group and U the endomorphism of X adjoint to T. It turns out that the ergodicity of an endomorphism T can be expressed in terms of the set-theoretical properties of U: *an endomorphism T is ergodic if and only if all trajectories of the transformation U, apart from that of the identity transformation, are infinite.* (see [44], [22], and [48]).

If T is ergodic then it satisfies stronger conditions in the theory of measure-preserving transformations. In [44], [22], [26] the following is proved: an ergodic endomorphism on a commutative group has a Lebesgue spectrum of countable multiplicity, is mixing of all degrees and has positive entropy. All these statements result from the more general theorem: *an ergodic endomorphism has completely positive entropy.* (see [28], [38]).

The following concepts play an important part in the proof of this theorem. Let F be the direct product of the sequence $\{H_i\}$ $(i = 0, \pm 1, \ldots)$ (infinite in both directions) of copies of a compact group H with a countable topological basis. The automorphism R of the group F defined by $R\{h_i\} = \{h_i'\}$, $h_i' = h_{i+1}$ $(h_i \in H_i)$, is called a *Bernoulli group automorphism with group of states H.* Bernoulli group automorphisms result, for example, from any ergodic automorphism on a connected compact group without centre (see [38]).

An automorphism T of G is said to be *densely periodic* if G contains an everywhere dense set A such that for any $a \in A$ there exists an integer n for which $T^n a = a$. Trivial examples of densely periodic automorphisms are the automorphisms of finite-dimensional tori and the Bernoulli group endomorphisms. Non-trivial is the proposition: *every ergodic automorphism of a totally disconnected compact group is densely periodic.* In the proof of this statement in [38] there is a more precise description ergodic automorphisms of a totally disconnected group: they are all skew products of Bernoulli group automorphisms.

Another subject in the metric theory of group endomorphisms is the calculation of their entropy. First we note the addition theorem: *If H is*

[1] A lecture given at the Khumsan school on ergodic theory.

a normal subgroup of G invariant under the endomorphism T, and if T_0 and S are the endomorphisms induced by T in H and G/H, then $h(T) = h(S) + h(T_0)$, where h is the entropy (see [38]). For the proof of this theorem and the extension of its domain of applicability we have to define the concept of entropy for an endomorphism that is not an epimorphism. This is trivial and is given in [38].

The calculation of the entropy of endomorphisms began almost as soon as the concept of the entropy of an automorphism appeared (see [30]). Computations of entropy are in the papers [3], [6], [8], [9], [40]. The most general result is contained in [40], where the entropy of an arbitrary endomorphism T of a connected commutative finite-dimensional group is calculated (the basic technical difficulties for this were overcome in [8]). The character group of such a group is torsion-free and of finite rank, and so the automorphism U for a fixed basis in X occurs as a matrix A with rational entries. We denote by $\lambda_1, \ldots, \lambda_r$ the eigenvalues of A and by s the common denominator of the moduli of the coefficients of the characteristic polynomial of this matrix. Then[1]

$$h(T) = \log s + \sum_{|\lambda| \geq 1} \log |\lambda_i|. \tag{h}$$

Formula (h) makes it possible to define the entropy of a group endomorphism without using measure theory concepts. This means that the calculation of the entropy of an endomorphism of a compact group becomes a problem in topological algebra. If G is commutative, then the definition of entropy becomes purely algebraic and is given as follows.

Let X be a commutative countable group and U a one-to-one endomorphism of X. We define the concept of algebraic entropy $h_a(U)$ of the endomorphism U.

a) Suppose first that X is periodic. Then we put

$$h_a(U) = \sup_{Y \in \mathfrak{A}} \log \operatorname{ord} \left(\bigvee_{n=0}^{\infty} U^n Y / \bigvee_{n=1}^{\infty} U^n Y \right),$$

where $\mathfrak{A}$ denotes the collection of finite subgroups of X.

b) Now we assume that X is a torsion-free group. If X contains an element x_0 such that for any polynomial p with integer coefficients $p(U)x_0 \neq 0$, then we put $h_a(U) = \infty$.

We assume for every $x \in X$ the existence of a polynomial p_x with integer coefficients such that $p_x(U)x = 0$. We enumerate the elements of X in a sequence $x_1, x_2, \ldots$ and denote by X_n the smallest subgroup of X containing $x_1, x_2, \ldots, x_n$ and invariant under U. The rank of X_n is finite, and so the endomorphism U_n induced by U in X_n can be expressed with respect to a fixed basis in X_n as a matrix A_n with rational entries. We denote by $\lambda_i^{(n)}$ the eigenvalues of A_n and by $s^{(n)}$ the common denominator of the moduli of the coefficients of its characteristic polynomial, and we put

$$h_a(U_n) = \log s^{(n)} + \sum_{|\lambda_i^{(n)}| \geq 1} \log |\lambda_i^{(n)}|.$$

[1] Here and below the logarithm is to the base 2.

The sequence $\{h_a(U_n)\}$ is increasing, and so the following makes sense: $h_a(U) = \lim_{n\to\infty} h_a(U_n)$.

c) Finally we consider the general case. We denote by Y the periodic part of the group X, by U_0 the endomorphism induced in Y by U, by V the factor-endomorphism of U in X/Y, and we put

$$h_a(U) = h_a(V) + h_a(U_0).$$

This definition gives the entropy of an endomorphism in the usual sense. To be precise, *if G is the character group of X and T is the endomorphism of G adjoint to U, then we have $h(T) = h_a(U)$* (see [40]).

If G is non-commutative, we denote by C the connected component of the identity, by Z the centre of C and by T_0, S and R the endomorphisms induced by T in Z, G/C and C/Z. Then the following are true (see [40]):

a) $h(T) = h(T_0) + h(S) + h(R)$,

b) $h(S) = \sup_{A \in \mathfrak{A}} \log \operatorname{ord} \left(\bigcap_{n=1}^{\infty} T^{-n}A / \bigcap_{n=0}^{\infty} T^{-n}A \right)$,

where $\mathfrak{A}$ denotes the collection of open normal subgroups of G/C.

c) *If R is the direct product of automorphisms of Lie groups, then $h(R) = 0$. Otherwise $h(R) = \infty$.*

Together with the algebraic definition of entropy for commutative groups these three statements give a topological-algebraic definition for the entropy of a group endomorphism in the general case.

In conclusion we consider automorphisms of an arbitrary locally-compact group. In this case T does not have to preserve measure. We call on automorphism T *ergodic* if for every measurable set A either $\mu(A) = 0$ or $\mu(G\setminus A) = 0$ and *weakly ergodic* if for any such A we have $\mu(A) = 0$ or $\mu(A) = \infty$. Clearly the conditions for T to be ergodic or weakly ergodic do not depend on the choice of the Haar measure on G.

All automorphisms that do not preserve measure are weakly ergodic. There exist weakly ergodic automorphisms that preserve measure. As an example we can take the linear transformation of the real plane given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

and also the direct product of S with any group automorphism. This is not a chance example: if a commutative group generated by a compact set, has a measure-preserving weakly ergodic automorphism, then the plane is a direct summand of it.

The fundamental theory in the metric theory of automorphisms of non-compact groups is: *an automorphism of a commutative or connected non-compact group is not ergodic* (part of this is proved in [39])[1]. It can be shown that a general solution depends on a solution for a non-compact totally disconnected group. Whether an ergodic automorphism exists for such a case is not known.

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[1] Added in proof. This theorem is also proved in the recently published papers [52], [53], [54].

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