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Abstract.

1. Counting Saddle Connections on a generic Translation Surface

Let $\mathcal{H}$ be the subset of area one surfaces inside a connected component of a stratum $\mathcal{H}(\alpha)$ of translation surfaces of fixed genus, fixed number of singular points, and fixed cone angle at each singular point determined by α. A saddle connection of a translation surface $x \in \mathcal{H}$ is a geodesic connecting two singular points without singular points in its interior. Its holonomy vector is the vector in the plane obtained by integrating along the horizontal and vertical foliation. Denote by $V(x)$ the collection of all holonomy vectors of x. Let $N(T,x) = |V(x) \cap B(T,0)|$ the number of holonomy vectors of length less than T. There is a natural smooth probability measure μ on $\mathcal{H}$, which is preserved by the $\mathrm{SL}_2(\mathbb{R})$-action on $\mathcal{H}$ obtained by postcomposition on charts of a translation surface (or acting on its representation as collection of polygons).

Theorem 1.1 ([EM01])**.** *There exists $c > 0$ such that for μ-a.e. $x \in \mathcal{H}$,*

$$N(T,x) = \pi c T^2 + o(T^2).$$

The constant c is called Siegel-Veech constant, and obtained by the following formula. One defines the Siegel-Veech transform that takes a function on the plane $f : \mathbb{R}^2 \to \mathbb{R}$ and returns a function on $\mathcal{H}$ by $\widehat{f}(x) = \sum_{v \in V(x)} f(v)$. Note that $N(T,x) = \widehat{\mathbb{1}_{B(T,0)}}(x)$.

Theorem 1.2 ([Vee98])**.** *There exists $c > 0$ such that for any $f \in L^1(\mathbb{R}^2)$*

$$\int \widehat{f}(x) d\mu(x) = c \int_{\mathbb{R}^2} f(y) dy$$

In particular, $\hat{f} \in L^1(\mathcal{H}, \mu)$.

This builds crucially on the fact that μ is ergodic with respect to $\mathrm{SL}_2(\mathbb{R})$ (independendly proven by Masur [Mas82] and Veech [Vee86]) and that there exists an upper bound $N(T,x) \leq c_x T^2$ for any translation surface $x \in \mathcal{H}$ (a result obtained by Masur).

A theorem of Howe-Moore implies that any $\mathrm{SL}_2(\mathbb{R})$-ergodic action is in fact mixing. This has been quantified by Avila-Gouëzel-Yoccoz. Recall that ϕ is an SO_2-eigenfunctions if for $k_\theta \in \mathrm{SO}_2$ a rotation corresponding to angle θ, one has $\phi(kx) = \exp(i\theta n)\phi(x)$ for some $n \in \mathbb{Z}$. Let $\|\cdot\|$ denote a matrix norm on $\mathrm{SL}_2(\mathbb{R})$.

Theorem 1.3 ([AGY06])**.** *There exists $\kappa > 0$ and $C > 0$ such that for any $g \in \mathrm{SL}_2(\mathbb{R})$ and any $\phi_1, \phi_2 \in L^2(\mathcal{H}, \mu)$ that are SO_2-eigenfunctions and not constant,*

$$\langle g \cdot \phi_1, \phi_2 \rangle_{L^2(\mathcal{H},\mu))} \leq C \|\phi_1\|_2 \|\phi_1\|_2 \|g\|^{-\kappa}.$$

In joint work with Amos Nevo and Barak Weiss we can deduce a polynomial error term for the counting problem.

Theorem 1.4 ([NRW])**.** *There exists $\kappa > 0$, $c > 0$ such that for μ-a.e. $x \in \mathcal{H}$,*

$$N(T,x) = cT^2 + O(T^{2-\kappa}).$$

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Moreover, one may can count for almost all x in any sector of any ellipse (note the order of quantifiers, the implicit constant in the $\mathcal{O}$-notation depending on these choices however). Equivalently, there exists a $\mathrm{SL}_2(\mathbb{R})$-invariant set of full measure for which the asymptotic holds for any sector. Without error rate, this also follows from a result of Veech in [Vee98] (under the assumption of a spectral gap and L^2-integrability).

This theorem holds in fact for almost all points on any $\mathrm{SL}_2(\mathbb{R})$-orbit closure in $\mathcal{H}$ with respect to its natural measure, whose existence is assured by work of Eskin, Mirzakhani and Mohammadi ([EMi15],[EMiMo15]). The corresponding spectral gap result were extended by Avila and Gouëzel [AG13].

Another crucial ingrediant is the following non-divergence estimate of Eskin-Masur. Let $a_t = \mathrm{diag}\,(e^t, e^{-t})$ and $K = \mathrm{SO}_2$. Define $K * a_t * f(x) = \int_K f(a_t kx)dk$ and let $\ell(x) = \min_{v\in V(x)} \|v\|$ the length of the shortest saddle connection.

Theorem 1.5. [EM01] *For $\delta \in [1, 2)$, for any $x \in \mathcal{H}$,*

$$\sup_{t>0} K * a_t * \ell^{-\delta}(x) < \infty.$$

One can deduce from this Theorem that $\ell^{-1} \in L^{1+\delta}(\mathcal{H}, \mu)$. The following recent result of Athreya-Cheung-Masur regarding square-integrability of the Siegel-Veech transform improves in a certain sense this estimate, and simplifies the proof of Theorem 1.4.

Theorem 1.6 ([ACM])**.** *For any $T > 0$*

$$N(T, \cdot) \in L^2(\mathcal{H}, \mu).$$

The counting proceeds by a geometric argument, first developed by Eskin-Margulis-Mozes and Eskin-Masur with a simplified proof (of a stronger statement) in [NRW]:

Lemma 1.7. *For any ε, there exists $\phi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}_{>0}$ satisfying that $\|\phi_\varepsilon\|_\infty = \mathcal{O}(\varepsilon^{-1/2})$, $\|\partial_\theta \phi_\varepsilon\|_\infty = \mathcal{O}(\varepsilon^{-3/2})$ and $\int_{\mathbb{R}^2} \phi_\varepsilon = 1$ such that*

$$\frac{1}{\pi e^{2t}} N(e^{2t}, x) = \left(1 + \mathcal{O}(\varepsilon^{1/2})\right) K * a_t * \widehat{\phi_\varepsilon}(x)$$

ϕ_ε is an ε-approximation of a sector of the unit ball with angle $\theta \asymp \varepsilon^{1/2}$. The key observation is that $a_t{\cdot}\phi_\varepsilon$ approximates now a sector of the ball of radius e^{2t}, and integrating at a vector v along the K-orbit, there is a contribution on a segment of length $e^{-2t}(\theta + 2\varepsilon)/2\pi$ iff v has length less than e^{2t}. Since we normalized ϕ_ε by dividing by the area $\asymp \theta$ one reads of the error as claimed.

The effective mixing statement may be phrased with respect to the first degree K-Sobolev norm $\mathcal{S}_K^2(f) = \|f\|_2^2 + \|\omega f\|_2^2$ on $L^2(\mathcal{H}, \mu)$ (where $\exp \mathbb{R}\omega = K$). We note that finiteness for $f = \widehat{\phi_\varepsilon}$ only follows after knowledge of Theorem 1.6. While the precise L^2-bound is unknown (by lack of a "Roger's formula"), we know that ϕ_ε are supported in a fixed compact set (say $B(2, 0)$), so that $\|\widehat{\phi_\varepsilon}\|_2 \leq \|\phi_\varepsilon\|_\infty \|N(2, x)\|_2 = \mathcal{O}(\varepsilon^{-1/2})$. $\mathrm{SL}_2(\mathbb{R})$-equivariance of the Siegel-Veech transform can be used to bound the ω-derivative, $\|\omega\widehat{\phi_\varepsilon}\|_2^2 = \mathcal{O}(\varepsilon^{-3})$.

If $k_\theta = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$, then the set of θ for which $\|a_t k_\theta a_{-t}\| \leq e^t$ is of (K-)measure $\mathcal{O}(e^{-t})$. Since $\|K * a_t * f\|_2^2 = \langle a_{-t} * K * a_t f, f\rangle$ we have, assuming $\|f\|_\infty \leq \mathcal{S}_K(f)$,

Lemma 1.8 ([Vee98],[EMaMo98])**.** *For any K-smooth $f \in L^2(\mathcal{H}, \mu)$,*

$$\|K * a_t * f - \mu(f)\|_2^2 \ll e^{-t\kappa} \mathcal{S}_K(f)^2.$$

By a Borel-Cantelli argument, one can obtain an effective pointwise ergodic theorem for $K * a_t * f(x)$ generic points x which together with Lemma 1.7 implies Theorem 1.4.

Another application of Lemma 1.7 and Lemma 1.8 is a bound for the variance of the counting function. Here, uniformity of the error term with respect to x in the former lemma is crucial:

$$\|N(T,\cdot)\|_2^2 = (1+\mathcal{O}(\varepsilon^{1/2}))\pi^2 e^{4t}\|K * a_t * \widehat{\phi_\varepsilon}\|_2^2.$$

Since $\|\widehat{f}-\mu(f)\|_2^2 = \|\widehat{f}\|^2 - \mu(f)^2$ and the Siegel-Veech formula, $\mu(\phi_\varepsilon) = c\int_{\mathbb{R}^2}\phi_\varepsilon = c$,

$$\|K * a_t * \widehat{\phi_\varepsilon}\|_2^2 - c^2 \leq \mathcal{O}(e^{-t\kappa}\varepsilon^{-6})$$

we see

$$\|N(T,\cdot)\|_2^2 = c^2\pi^2 e^{4t} + \mathcal{O}(e^{4t-\kappa t}\varepsilon^{-5.5} + e^{4t}\varepsilon^{1/2}).$$

Taking ε to be an sufficiently small power of e^{-t} one has proven the following.

Theorem 1.9 (Athreya-R)**.** *There exists $\kappa' > 0$ for which*

$$\|N(e^t,\cdot)\|_2 = \pi c e^{2t} + \mathcal{O}(e^{(2-\kappa')t}).$$

We note that this theorem implies Theorem 1.4, see [ACM].

The same argument gives also variation estimates for orbits of non-uniform lattices $\Gamma < \mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{R}^2$. As for the required inputs: It is known that any finite volume quotient $\mathrm{SL}_2(\mathbb{R})/\Gamma$ exhibits a spectral gap (i.e. is weakly compact in the sense of Margulis). Moreover, $N(T,x)$ is a bounded function on $\mathrm{SL}_2(\mathbb{R})/\Gamma$, see [Vee98], Lemma 16.10.

Theorem 1.10. *Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a non-uniform lattice, $v \in \mathbb{R}^2 \setminus \{0\}$ such that Γv is discrete. For $x = g\Gamma \in \mathrm{SL}_2(\mathbb{R})/\Gamma$ let $N(T,x) = |g\Gamma v \cap B(T,0)|$ then*

$$\|N(T,\cdot)\|_2 = c_\Gamma T^2 + \mathcal{O}(T^{2-\kappa_\Gamma})$$

Burton Randol has a much stronger result for the special case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, showing that $\|N(T,\cdot)\|_2^2 = \zeta(2)^{-1}\pi T^4 + \zeta(2)^{-1}\pi T^2 + \mathcal{O}(T^2\log^{-k} T)$ for any integer k.

In [CNRW] we are quantifying[1] Veech's Eisenstein approach [Vee89] for pointwise estimates of $N(T,x)$ (and generalizing it to sectors) which are valid for any x, and deal with more general shapes building on the lattice point techniques of Gorodnik-Nevo.

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[1] Using estimates of Good, Veech already obtained $T^{4/3}$ rates of the ball counting problem for tempered quotients