Furstenberg Thm

m_0 prob on $SL_d(\mathbb{R})$

s.t.
- $\operatorname{supp} m_0$ leaves no finite set of subspaces of $\mathbb{R}^d$ invariant
- $\operatorname{supp} m_0 \not\subset$ cmpct group

$\Longrightarrow$ $\exists$ no prob on $\mathbb{P}^{d-1}$ inv under $\langle \operatorname{supp} m_0 \rangle$

$\Longrightarrow$ $\lambda^+ > \lambda^-$ $\qquad \lambda^{\pm} = \lim_{n\to\infty} \frac{1}{\pm n} \int \log \|g\| \, dm_0^{\otimes n}(g)$

Example

$m_0 = \frac{1}{2}\left(\delta_{\begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}} + \delta_{\begin{bmatrix} \text{irr} \\ \text{rat} \end{bmatrix}}\right)$

$\overline{\langle \text{rat} \rangle} = K$

$\langle a \rangle \leq A$

Skew product construction:

Let ν_0 Prob on $\mathbb{P}^{d-1}$ be m_0-stationary: $m_0 * \nu_0 = \nu_0$
$$= \int g * \nu_0 \, dm_0(g)$$

$p = m_0^{\mathbb{N}}$ $\qquad X = \{\operatorname{supp} m_0\}^{\mathbb{N}}$ $\qquad T$ shift map

$x = (\omega_0, \omega_1, \omega_2, \dots)$ $\qquad X \to SL_d(\mathbb{R})$, $\quad x \mapsto A_x = \omega_0$

$\hat{T} : X \times \mathbb{P}^{d-1} = \hat{X} \longrightarrow \hat{X}$ $\qquad \nu = p \otimes \nu_0$ is $\hat{T}$-inv

$\hat{T}(x, s) = (Tx, A_x s)$

(More general setup: (X, p), $\pi_X * \nu = p$, $\nu = \int \nu_x \, dp$)

$\Omega = (\operatorname{supp} m_0)^{\mathbb{Z}}$ $\qquad \Theta$ shift map $\qquad m = m_0^{\otimes \mathbb{Z}}$

$\omega = (\dots \omega_{-1}, \omega_0, \omega_1, \dots)$ $\qquad \pi : \Omega \to X$ $\qquad \pi_* m = p$

Construct $\mu = \int \mu_\omega \, dm$

by defining $\mu_\omega := \lim_{n\to\infty} \omega_{-1} \cdots \omega_{-n} * \nu_0$

More general: $\mu_\omega = \lim_{n\to\infty} A^n_{\Theta^{-n}\omega} * \nu_{\pi(\Theta^{-n}\omega)}$

Exist by Martingal Convergence Thm $\qquad \Theta^n \mathcal{B}_X \nearrow \mathcal{B}_\Omega$

By construction $\mu_{\theta(\omega)} = A_\omega * \mu_\omega$, $\hat{\pi}_* \mu = \nu$

$\Updownarrow$ (by Θ, T-inv)

μ is $\hat{\Theta}$-inv

Last time:

$$\begin{array}{ccc}
(\hat{\Omega}, \mu) & \xrightarrow{\hat{\Theta}} & (\hat{\Omega}, \mu) \\
\downarrow & & \downarrow \\
(\hat{X}, \nu) & \xrightarrow{\hat{T}} & (\hat{X}, \nu)
\end{array}$$

decreasing $\mathcal{A} = \pi^{-1}(\mathcal{B}_X)$ assume $\mathcal{A}_{-\infty} = \sigma\left(\bigvee \Theta^n \mathcal{A}\right) = \mathcal{B}_\Omega$

A $\mathcal{A}$-meas

Thm A: $\lambda^+ = \lambda^- \Rightarrow \omega \mapsto \mu_\omega$ is $\mathcal{A}$-meas

Furstenberg Thm followed by applying Thm to $(\hat{\Omega}, \hat{\Theta}^{-1} = (\Theta^{-1}, A^{-1}_{\Theta^{-1}\omega}))$ $\tilde{\mathcal{A}} = (\operatorname{supp} m_0)^{\mathbb{Z}_{<0}}$

$\perp \mathcal{A}$

$\Rightarrow \mu_\omega \equiv \nu$ const $\lightning$

Thm B: $J(x,s) = \dfrac{d A_x^{-1} * \nu_{Tx}}{d\nu_x}$

$$a_\nu(\hat{T}) = -\int \log J \, d\nu \leq (d-1)(\lambda^+ - \lambda^-)$$

Thm B $\Rightarrow$ Thm A:

$\Rightarrow a(\nu) = 0 \iff J \equiv 1 \iff \nu_{Tx} = A_x * \nu_x$

$\Rightarrow \mu_\omega = \lim_n A^n_{\pi\Theta^{-n}\omega} * \nu_{\pi\Theta^{-n}\omega} = \lim_n \nu_{\pi\omega} = \nu_{\pi(\omega)}$

is $\mathcal{A}$ meas

Will assume $a_\nu < \infty$ (true a fortiori)

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/ Geometric Input $\phi(A) := \frac{1}{\|A\|\|A^{-1}\|}$

\ Ergodic Thms (Assume $\log\|A_\omega^{\pm}\| \in L^1$)

<u>Lemma</u>: $\forall \varepsilon\ \exists \delta_0(\varepsilon)\ \forall \delta \leq \delta_0(\varepsilon)\quad \forall s \in \mathbb{P}^{d-1}\quad \forall A \in SL_d(\mathbb{R})$

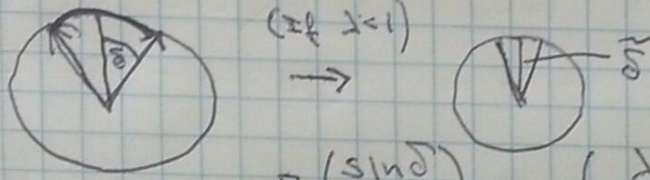
$$A\,B(s,\delta) \supset B(As, \delta\,\phi(A)e^{-\varepsilon})$$

<u>Proof</u>: $d=2$. $SL_2 = K \begin{bmatrix} * & \\ & * \end{bmatrix} K$ $\quad K \curvearrowright \mathbb{P}^{d-1}$ isometry

$(k B(s) = B(ks))$

may assume $A = \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix}$

$B(\uparrow, \delta)$



$$A\begin{pmatrix} \sin\delta \\ \cos\delta \end{pmatrix} = \begin{pmatrix} \lambda \sin\delta \\ \lambda^{-1}\cos\delta \end{pmatrix} = \begin{pmatrix} \sin\tilde{\delta} \\ \cos\tilde{\delta} \end{pmatrix}$$

$$\tilde{\delta} = \arctan(\lambda^2 \tan\delta)$$

$$\underset{1-\varepsilon}{\approx} \lambda^2 \delta = \phi(A)\delta$$

<u>Besicovitch Covering Lemma</u>

$\exists C > 0$ st $\forall E \subset \mathbb{P}^{d-1}$, $r : E \to \mathbb{R}_{>0}$

the cover $\{B(s, r_s) : s \in E\}$ of E

has a subcover st all pts are covered

<u>at most</u> C times

<u>Cor</u>: $\forall\ \nu_0, \tilde{\nu}_0$ prob on $\mathbb{P}^{d-1}$ ~~st~~

- $\overline{\lim_{\delta\to 0}} \frac{\log \nu_0(B(s,\delta))}{\log\delta} \leq d-1 \quad \nu_0$ ae

- $\lim_{\delta \to 0} \frac{\tilde{\nu}_0(B(s,\delta))}{\nu_0(B(s,\delta))} = \frac{d\tilde{\nu}_0}{d\nu_0}(s)$
- $\int \log \sup_\delta \frac{\tilde{\nu}_0(B(s,\delta))}{\nu_0(B(s,\delta))}\, d\nu_0 < C$

Proposition 3.1 $\exists\, h_\varepsilon(x,s)$ st (mean)

- If $\delta < h_\varepsilon(x,s)$ $\dfrac{\nu_{Tx}(B(A_x s, \delta\phi(A)e^{-\varepsilon}))}{\nu_x(B(s,\delta))} \le e^{\varepsilon} J(x,s)$
- If $\delta < \delta_0(\varepsilon)$ $\le J^*(x,s)$

$\overset{!!}{}$ $\sup_\delta \dfrac{\nu_{Tx}(AB)}{\nu_x(B)}$

$\delta \le \delta_0$

$\Gamma \xleftarrow{\text{No!}}$ as $\delta \to 0$ $LHS \le \dfrac{A_x^{-1}\nu_{Tx}(B)}{\nu_x(B)} \to J$

$\le J e^{\varepsilon}$

always have 2^{nd}. $\le J^*$

(dep on x,s)

~~$h_\varepsilon \to 0$~~

~~$h_\varepsilon \le \delta_0$~~

$\perp$ $h_\varepsilon < \delta$

Proposition (Assume ν ergodic)

ν ae (x,s)
$$\varliminf_n -\frac{1}{n}\log \nu_{T^n x}\Big(B\big(A_x^n s, \delta_0 e^{-n\varepsilon}\prod_{i<n}\phi(A_{T^i x})\big)\Big)$$
$$\geq a_\nu - 3\varepsilon$$

Proof: Telescope:
$$\nu_{T^n x}(\ -n-\ ) = \nu_x(B(s,\delta_0))\prod_{j<n} q_j(x,s)$$
$$q_j(x,s) = \frac{\nu_{T^{j+1}x}(\ -j+1-\ )}{\nu_{T^j x}(\ -j-\ )}$$

$\forall j \ge N$ suff large $x_j = \delta_0 e^{-j\varepsilon}\prod \phi(A_{T^i x}) < \delta < \delta_0$

Note $x_{j+1} = x_j e^{-\varepsilon}\phi(A_{T^j x})$

~~so $q_j(x,s)$~~

Apply prev prop to the pt $\hat{T}^j(x,s)$ gives
$$q_j(x,s) \le e^{\varepsilon} J(T^j(x,s)) \quad (\text{if } \delta < h_\varepsilon \circ T^j(x,s))$$
$$\le J^* \circ T^j(x,s) \quad \text{else}$$

Define $g(x,s)$ by $\begin{cases} = \cancel{J} \\ = J^* \end{cases}$

$\to \log \prod_{j<n} g_j \leq \log \prod_{j<N} g_j + n\varepsilon + \sum_{j\geq N}^{n} \log g \circ T^j$

Apply $-\frac{1}{n}\lim$ $\quad \geq -\varepsilon + \int \log g \, d\nu$

$= -\varepsilon + \underbrace{\int_{J<h_\varepsilon} \log J}_{\geq a_\nu - \varepsilon} + \underbrace{\int_{h_\varepsilon<\delta} \log J^*}_{\leq \varepsilon \; (J^* \geq 1)}$

choose δ small s.t.

$\geq a_\nu - 3\varepsilon$ $\blacksquare$

Proof of Thm B $\quad \left(a_\nu \leq (d-1)(\lambda^+ - \lambda^-) \right)$

Follows from $\bigstar$ $a_\nu \leq (d-1)\, \mathbb{E}[-\log \phi(A_\cdot)]$

$\ulcorner$ Apply $\bigstar$ to the system $(\nu, \hat{T}^n)$

and use $a_\nu(\hat{T}^n) = n\, a_\nu(T)$

(chain rule Jacobian)

so $a_\nu \leq (d-1) \lim_n \frac{1}{n} \mathbb{E}[\log \|A^n\| + \log \|(A^n)^{-1}\|]$

$= (d-1)(\lambda^+ - \lambda^-)$ $\perp$

By ergodic decomp suffices to proof $\bigstar$ w/ ν ergodic (?!)

Prop: ν ergodic $\bullet\ \dim \nu \le d-1$ (to be def)

$\bullet\ a_\nu \le \dim \nu \ \mathbb{E}[-\log \phi(A)]$

$$\dim \nu = \lim_{x \to 0} \beta(x)$$

$$\beta(x) = \sup\left\{ t : \exists \varepsilon_t \ \forall \varepsilon < \varepsilon_t \quad \nu\left(\nu_x(B(s,\varepsilon)) < \varepsilon^t \right) \ge 1-x \right\}$$

Proof: $\dim \nu \le d-1$ by BCL

By Erg Thm $\nu(D_1) \ge 1 - \frac{x}{2}$ (n large)

$$D_1 = \left\{ (x,s) : \prod_{j<n} \phi(A_{T^j s}) \ge e^{n(\mathbb{E}\log\phi - \varepsilon)} \right\}$$

Plug in prop about $\lim -\frac{1}{n}\log \nu_{T^n x}(\quad)$:

$$\nu(D_2) \ge 1 - x$$

$$D_2 = \left\{ (x,s) : -\frac{1}{n}\log \nu_{T^n x}\left(B\left(A^n_x s, \delta_0 e^{-2n} e^{n\mathbb{E}\log\phi}\right)\right) \ge a_\nu - 3\varepsilon \right\}$$

Can replace $(T^n x, A^n_x s)$ by $\hat{T}$-invariance

$$\beta(x) \ge \frac{(a_\nu - 3\varepsilon)n}{\left(n\mathbb{E}[\log\phi] - 2\varepsilon n + \log\delta_0\right)} = \frac{a_\nu - 3\varepsilon}{-\mathbb{E}[\log\phi] + 2\varepsilon - \frac{\log\delta_0}{n}}$$

$n \to \infty$

$\varepsilon \to 0$ ∎