Lecture #9

Today: Calculate the entropy

G/Γ

$a = \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} \curvearrowright G/\Gamma$

$\begin{bmatrix} 1 & s \\ & 1 \end{bmatrix} \circlearrowright$

$A \curvearrowright \mathbb{T}^d \qquad A \in SL_d(\mathbb{R})$

EL - Pisa notes

EW - homogeneous book

$G_a^- = \{ g \in G : a^n g a^{-n} \to e \}$

a-inv prob μ on $X = G/\Gamma$

$H = G_a^- \curvearrowright X$ would like to restrict μ to H-orbits

Lemma: (X, T, μ) μ T-erg invertible

$\mathcal{E} = \sigma(A \in \mathcal{B}_X : T^{-1}A = A)$

$\mathcal{E}$ c.g. $\Rightarrow$ μ is supported on a finite set

Proof: $\forall A \in \mathcal{E} \quad \mu(A) \in \{0, 1\}$

$\mathcal{E} = \sigma\{E_n\}$

$x \in \bigcap_{\mu(E_n)=1} E_n \cap \bigcap_{\mu(E_n)=0} E_n^c = [x]_{\mathcal{E}} = \{T^n x\}_{n \in \mathbb{Z}}$

$\rightsquigarrow$ finite orbit since $\mu([x]_{\mathcal{E}}) = 1$ ∎

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cond. measures $\rightsquigarrow$ fibre measure $\rightsquigarrow$ leafwise measures
(of finite measures) (locally finite measures)

<u>Note</u>: • μ finite measure, $\mathcal{A}$

$$\mu = \int \mu_x^{\mathcal{A}} \, d\mu(x)$$

$\uparrow$ always probabilities

• μ only finite $\forall Y \subset X$ $\mu(Y) < \infty$

$$\mu|_Y = \int (\mu|_Y)_x^{\mathcal{A}} \, d\mu(x)$$

X σ-compact $X = \bigsqcup Y_n$

<u>Fibre measures</u>: X σ-compact, loc compact, μ loc finite

$\mathcal{A} \subset \mathcal{B}_X$

Fix $f_0 \in L^1(\mu)$ $f_0 > 0$ μ a.e. $\mu(f) = 1$

Define ν on X ν_{f_0} defined by $d\nu_{f_0} = f_0 \, d\nu$

μ_{f_0} prob on X

$$\mu_x^{\mathcal{A}} := \left((\mu_{f_0})_x^{\mathcal{A}} \right)_{\frac{1}{f_0}}$$

Prop:

- $x \mapsto \mu_x^{\mathcal{A}}$ $\mathcal{A}$-measurable ✓
- $\forall Y$ $\mu(Y) < \infty$

standard cond. prob.

$$\mu_x^{\mathcal{A}}|_Y \sim (\mu|_Y)_x^{\mathcal{A}}$$

up to proportionality

$$\ulcorner \; \frac{\mu_x^{\mathcal{A}}|_Y}{\mu_x^{\mathcal{A}}(Y)} = (\mu|_Y)_x^{\mathcal{A}} \; \lrcorner \quad (*)$$

Proof: $g \in L^\infty(Y)$ $\quad A \in \mathcal{A}|_Y$

$$(*) \iff \int_A \frac{\mu_x^{\mathcal{A}}(g)}{\mu_x^{\mathcal{A}}(Y)} \frac{\mathbb{1}_Y}{f_0} \, d\mu_{f_0} \overset{!}{=} \int_A g \, d\mu$$

$$\| $$

$$\int_A \frac{\mu_x^{\mathcal{A}}(g)}{\cancel{\mu_x^{\mathcal{A}}(Y)}} \underbrace{(\mu_{f_0})_x^{\mathcal{A}} \left(\frac{\mathbb{1}_Y}{f_0} \right)}_{= \cancel{\mu_x^{\mathcal{A}}(\mathbb{1}_Y)}} \, d\mu_{f_0}$$

$$= \int_A \mu_x^A(g)\, d\mu_{f_0} = \int_A \frac{g}{f_0}\, d\mu f_0$$

$$= \int_A g\, d\mu \qquad \blacksquare$$

Transformation rule for fibre measures:

$T: X \to X \quad T_*\mu = \mu$ then $T_* \mu_x^{T^{-1}A} \sim \mu_{Tx}^A$

$$\left[\begin{cases} \phi : (X,\mu) \to (Y,\nu) \\ \quad \phi_*\mu = \nu \\ \mu,\nu \text{ finite} \end{cases}\right. \quad A \subset \mathcal{B}_Y \quad \text{then} \quad \phi_* \mu_x^{\phi^{-1}A} = \nu_{\phi(x)}^A$$

Apply: Take $Y \subset X \quad \mu(Y) < \infty$

$$T : T^{-1}Y \to Y \qquad \text{setting} \begin{cases} \\ \end{cases}$$

$$\mu|_{T^{-1}Y} \to \mu|_Y$$

$$T_* \left(\mu|_{T^{-1}Y}\right)_x^{T^{-1}A} \overset{?}{=} \left(\mu|_Y\right)_{Tx}^A$$

$$\underset{\displaystyle T_*\mu_x^{T^{-1}A}\Big|_Y = T_*\left(\mu_x^{T^{-1}A}\Big|_{T^{-1}Y}\right)}{\int} \qquad \underset{\displaystyle \mu_{Tx}^A\Big|_Y}{\text{"}} \qquad \blacksquare$$

Leafwise measure: μ loc finite on G/Γ

$H < G$ closed

$H\backslash G$ $\quad \mathcal{B}_{H\backslash G}$ c.g.

Lift μ to μ_G on G Γ-invariant.

$$\mu_g = (\mu_G)_g^{\mathcal{B}_{H\backslash G}}$$

fibre measure support on Hg

Leafwise measure:

$x = g\Gamma \in G/\Gamma$

right translation

defined μ a.e. $x \in G/\Gamma$

$$\mu_x^H = R_{g^{-1}} * \mu_g =: \mu_g * g^{-1}$$

supported on H

$\gamma \in \Gamma$ $\quad R_\gamma * \mathcal{B}_{H\backslash G} = \mathcal{B}_{H\backslash G}$

$$\mu_g * \gamma \sim \mu_{g\gamma}$$

well defined up to proportionality

$$\mu_{g\gamma} * (g\gamma)^{-1} = \mu_{g\gamma} * \gamma^{-1} * g^{-1}$$

$$\sim \mu_g * g^{-1}$$

we can fix a normalization:

$$\mu_x^H(B_1^H(e)) = 1$$

Lemma: $e \in \operatorname{supp} \mu_x^H$ $\quad \mu$ ae x

Γ

$$\mu_g(Y) > 0 \quad \text{ae } g \in Y$$

✓

$\mu_G(Y) < \infty$

$\{g_k\} \subset G$ dense

$$\mu_g(B_{\frac{1}{n}}(g_k)) > 0$$

$\Rightarrow$

$$\mu_g(B^H_{\frac{2}{n}}(g^{-1}g)) > 0 \quad \text{ae } g \in B_{\frac{1}{n}}(g_k)$$

$\forall n$ $\quad \supset B^H_{\frac{1}{10}\frac{1}{n}}(e)$

indep n, k

$\perp$

Lemma: $a \in N_G(H)$, $a * \mu = \mu$

$$\mu^H_{a.x} \sim \operatorname{Conj}(a) * \mu^H_x$$

Γ $\quad a^{-1} B_{H|G} = B_{H|G} \rightsquigarrow a * \mu_g \sim \mu_{a.g}$

$x = g\Gamma$ $\quad \mu^H_{a.x} \sim \mu_{ag} * g^{-1}a^{-1} \sim a * \mu_g * g^{-1} * a^{-1}$

$\sim a * \mu_x^H * a^{-1}$

$\perp$

- $a \in SL_d(\mathbb{R})$ diagonal

  $H < G_a^- = \{ g : a^n g a^{-n} \to e \}$

  H a-normalized

- $A \curvearrowright \mathbb{T}^d$

  $G = \mathbb{Z} \ltimes \mathbb{R}^d$

  $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^d$

  $(m, v) \cdot (n, w)$

  $= (m + n, A^m w + v)$

  $a = (1, 0) \mathrel{\hat{=}} \underline{A}$

  $G_a^- \mathrel{\hat{=}}$ Eigenspaces for $|\lambda| < 1$

Thm: μ H-inv $\iff$ μ_x^H left Haar a.e. x

Proof:

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

μ_G H-inv $\overset{!}{\iff}$ μ_g left H-inv a.e. g

$[\Rightarrow]$: $h_* \mu_G = \mu_G \qquad h_* B_{H\backslash G} = B_{H\backslash G}$

$$\Rightarrow \quad h_* \mu_g \sim \mu_{hg} = \mu_g$$

$$\exists\, c_g \quad h_* \mu_g = c_g \mu_g$$

c_g H-inv.

$B' \subset B = \{ c_g > 1 \}$

finite measure

$Y = B' \sqcup h^{-1} B'$

$\mu_G(Y) < \infty$

$c_g > 1$ on $g \in Y$

$$\mu_G|_Y (B') = \int \frac{\mu_g(B')}{\mu_g(Y)} \, d\mu_G|_Y$$

$$= \int \frac{1}{c_g} \frac{\mu_g(h^{-1}B')}{\mu_g(Y)} \, d\mu_G|_Y$$

$$< \int \frac{\mu_g(h^{-1}B')}{\mu_g(Y)} \, d\mu_G|_Y$$

$$= \mu_G|_Y (h^{-1}B') \quad ⚡$$

$\Rightarrow \mu_G(B) = 0$

$\Rightarrow c_g \equiv 1$

[$\Leftarrow$] $B \subset G \quad Y = B \sqcup hB$

$$\mu_G|_Y(B) = \int \frac{\mu_g(B)}{\mu_g(Y)} \, d\mu_G(g)$$

$$= \int \frac{\mu_g(hB)}{\mu_g(Y)} \, d\mu_G(g)$$

$$= \mu_G(hB)$$

$H = U < G_a^-$ a-normalized

Def: Entropy contribution of U at x wrt a-inv prob μ on G/Γ

$$\mathrm{vol}_\mu^U(a)(x) = \lim_{n\to\infty} \frac{1}{n} \log \mu_x^U \left(a^{-n} B_1^U a^n\right)$$

Prop: $\mathrm{vol}_\mu^U(a)(x) \geq 0$, a-inv

$$= \lim_{n\to\infty} -\frac{1}{n} \log \mu_x^U \left(a^n B_1 a^{-n}\right)$$

and could replace B_1^U by Ω open, bnd

Proof: $\mu_x^U(B_1^U) = 1$

$$f(x) = \log \mu_x^U \left(a^{-1} B_1^U a\right) \geq 0$$

$$f(a^k x) = \log \frac{\mu_{a^k x}^U\left(a^{-1} B_1^U a\right)}{\mu_{a^k x}^U\left(B_1^U\right)}$$

$K \in \mathbb{Z}$

$$\overset{\text{Lemma}}{=} \log \frac{\mu_x^U\left(a^{-(k+1)} B_1^U a^{(k+1)}\right)}{\quad}$$

$$\mu_x^u(a^{-k} B_1^u a^k)$$

$$\frac{1}{n} \sum_{k \le n-1} f(a^k x) = \frac{1}{n} \log \mu_x^u (a^{-n} B_1^u a^n)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\mathbb{E}[f \mid \mathcal{E}](x) \geq 0 \qquad \mathrm{vol}_\mu^u(a)(x)$$

$\uparrow$ σ-alg a-inv sets

Example: $\mu = \text{Haar} = m \qquad \mu_x^u \equiv m_u$

$$m_u(a B_1 a^{-1}) = \mathrm{mod}_u(a)\, m_u(B_1)$$

$$\mathrm{mod}_u(a) = \left| \det \mathrm{Ad}_a|_{\mathrm{Lie}\,U} \right|$$

- $a = \begin{bmatrix} e^{-t} & \\ & e^t \end{bmatrix} \qquad U = \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix}$

$$\mathrm{vol}_\mu^u(a)(x) \equiv 2t$$

- $a = u = \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix} \qquad G_a^- = \{e\}$

$$\mathrm{vol}_\mu^{G_a^-}(a) = 0$$

- $\exists$ hyperbolic automorphism $V^- =$ contracting subgroup

$$\mathrm{mod}_{V^-}(a) = \prod_{|\lambda|<1} |\lambda|$$

$$\mathrm{vol}_\mu^{G_a^-}(a)(x) = \sum_\lambda \log^+|\lambda|$$

Thm A: $\mu = \int \mu_x^{\mathcal{E}} \, d\mu(x)$ erg decomp

Then $\mathrm{vol}_\mu^{U}(a)(x) \leq h_{\mu_x^{\mathcal{E}}}(a)$

"=" if $U = G_a^-$

Def: $h_\mu(a, U) = \int \mathrm{vol}_\mu^U(a)(x) \, d\mu(x)$

$\Rightarrow h_\mu(a, G_a^-) = h_\mu(a)$

Thm B: $h_\mu(a, U) \leq -\log \mathrm{mod}_U(a)$

"=" iff μ is U-invariant

Rmk: Formulate analogues for G_a^+
$G_a^- = G_{a^{-1}}^+$, $h_\mu(a) = h_\mu(a^{-1})$

Cor: • ~~&~~ $G = \langle G_a^-, G_a^+ \rangle$

then Haar is the unique measure of maximal entropy

• μ is erg and G_a^- inv $\Rightarrow$ G_a^+ inv ✓