<u>Lecture #8</u>

$(X, \mu, \mathcal{B}, T)$

Extremal values

Zero entropy proposition

$\mathcal{C}, \mathcal{A} \subset \mathcal{B}$ (countably gen.)

$$H_\mu(\mathcal{C}|\mathcal{A}) = 0 \iff \mathcal{C} \subset \mathcal{A} \ \text{mod } \mu$$

$$\| $$

$$\int -\log \mu_x^{\mathcal{A}}([x]_{\mathcal{C}})\, d\mu(x)$$

$\exists N$ null set $\forall C \in \mathcal{C}\ \exists A$ $C \setminus N = A \setminus N$

Proof: $[\Leftarrow]$ $\mu_x^{\mathcal{A}}([x]_{\mathcal{C}}) \geq \mu_x^{\mathcal{A}}([x]_{\mathcal{A}} \setminus N)$

$= 1$

$[\Rightarrow]$

$$\mu_x^{\mathcal{A}}([x]_{\mathcal{C}}) = 1 \quad \text{ae}$$

$\forall C \in \mathcal{C}$ $\quad \mu_x^{\mathcal{A}}(C) = \begin{cases} 1 & x \in C \\ 0 & x \in C^c \end{cases} = \mathbb{1}_C$

$\| $

$\mathbb{E}[\mathbb{1}_C | \mathcal{A}]$

$\Rightarrow C \in$

$\mathcal{A} \perp \mathcal{C}$ (independent) if $\mu(A \cap C) = \mu(A)\mu(C)$ $\forall A \in \mathcal{A}$, $C \in \mathcal{C}$

$\xi \perp \mathcal{C}$ $\quad \sigma(\xi) \perp \mathcal{C}$

$A \perp \mathcal{C}$ $\quad \{A, A^c\} \perp \mathcal{C}$

Maximal entropy proposition

ξ countable, $H_\mu(\xi) < \infty$, $\mathcal{C}$ c.g.

i) $A \perp \mathcal{C} \iff \mu_x^{\mathcal{C}}(A) = \mu(A)$ ae x

ii) $\xi \perp \mathcal{C} \iff H_\mu(\xi|\mathcal{C}) = H_\mu(\xi)$

Proof: i) $[\Leftarrow]$ $\mu(A \cap C) = \int_C \mathbb{1}_A = \int_C \mu_x^{\mathcal{C}}(A)\, d\mu(x)$
$C \in \mathcal{C}$

$$= \mu(A)\mu(C)$$

$[\Rightarrow]$ $x \mapsto \mu(A)$ $\mathcal{C}$-meas

$\forall C \in \mathcal{C}$ $\quad \int_C \mu(A) = \mu(A)\mu(C) = \mu(A \cap C)$

$$= \int_C \mathbb{1}_A$$

$$\Rightarrow \mu(A) = \mathbb{E}[\mathbb{1}_A | \mathcal{C}] = \mu_x^{\mathcal{C}}(A)$$

ii) $H_\mu(\xi|\mathcal{C}) = \int H_{\mu_x^{\mathcal{C}}}(\xi)\, d\mu$

$\phi(t) = -t\log t$ $\quad \bigcap$ convex

$$= \int \sum_{A \in \xi} \phi(\mu_x^{\mathcal{C}}(A)) \, d\mu(x)$$

$$\leq \sum_{A} \phi(\mu(A)) = H_\mu(\xi)$$

"=" iff $\mu_x^{\mathcal{C}}(A) \equiv \mu(A)$ ae x
$\forall A \in \xi$

Apply c) $\blacksquare$

Important Thms $\boxed{T \text{ invertible}}$

Additivity Proposition

ξ, η countable finite entropy $\quad \mathcal{A} = T^{-1}\mathcal{A}$ σ-alg

$$h_\mu(T, \xi \vee \eta \mid \mathcal{A}) = h_\mu(T, \xi \mid \mathcal{A}) + h_\mu(T, \eta \mid \xi_{-\infty}^{\infty} \vee \mathcal{A})$$

Proof:

$$\text{LHS} = \lim_n \frac{1}{n} H_\mu\left((\xi \vee \eta)_0^n \mid \mathcal{A}\right)$$

$$= \lim \frac{1}{n}\left[H_\mu(\xi_0^n \mid \mathcal{A}) + H_\mu(\eta_0^n \mid \xi_0^n \vee \mathcal{A})\right]$$

Split $\eta_0^n = \eta_1^n \vee \eta$

Rpt $\eta_1^n = \eta_2^n \vee T^{-1}\eta$

$$= h_\mu(T, \xi \mid \mathcal{A}) + \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n} H_\mu(T^{-j}\eta \mid \eta_{j+1}^n \vee \xi_0^n \vee \mathcal{A})$$

$$= h_\mu(T, \xi \mid \mathcal{A}) + \lim \frac{1}{n}\sum_j H_\mu(\eta \mid \eta_1^{n-j} \vee \xi_{-j}^{n-j} \vee \mathcal{A})$$

$$\|$$

Claim $\quad H_\mu(\eta \mid \eta_1^\infty \vee \xi_{-\infty}^\infty \vee \mathcal{A})$

[$\geq$] ok by monotonicity

$$H_\mu(\xi \mid \mathcal{A} \vee \mathcal{C}) \leq H_\mu(\xi \mid \mathcal{A})$$
$$\leq H_\mu(\xi)$$

[$\leq$] $\forall \varepsilon \exists N$

$$H_\mu(\eta \mid \eta_1^N \vee \xi_{-N}^N \vee \mathcal{A})$$

$$\leq H_\mu(\eta \mid \eta_1^\infty \vee \xi_{-\infty}^\infty \vee \mathcal{A}) + \varepsilon$$

by continuity

$$H_\mu\left(\eta \,\middle|\, \eta_1^{\overset{\geq N}{n-j}} \vee \xi_{\underset{\leq -N}{-j}}^{\overset{\geq N}{n-j}} \vee \mathcal{A}\right)$$

$j \in [N, n-N]$
$$\leq H_\mu\left(\eta \,\middle|\, \eta_1^{N} \vee \xi_{-N}^{N} \vee \mathcal{A}\right)$$

(Monotonicity)

$n \to \infty$

$\varepsilon \to 0$ $\blacksquare$

Thm: (Abramov-Rokhlin formula)

$$\begin{tikzcd}
(X, \mu) \arrow[r, "T"] \arrow[d, "\pi"'] & (X, \mu) \arrow[d, "\pi"] \\
(Y, \nu) \arrow[r, "S"] & (Y, \nu)
\end{tikzcd}$$

T, S invertible

$$\mathcal{A} = \pi^{-1}\mathcal{B}_Y \subset \mathcal{B}_X \qquad \mathcal{A} = T^{-1}\mathcal{A}$$

Then
$$h_\mu(T) = h_\nu(S) + h_\mu(T \mid \mathcal{A})$$

Proof: Recall Kolmogorov-Sinai thm

$\xi_n \nearrow \mathcal{B}_X$ $\quad h_\mu(T, \xi_n \mid \mathcal{C}) \to h_\mu(T \mid \mathcal{C})$

$\eta_m \nearrow \mathcal{B}_Y \quad \xi_n \nearrow \mathcal{B}_X$

Prev. prop

$$h_\mu(T, \xi_n \vee \pi^{-1}\eta_m) = h_\mu(T, \pi^{-1}\eta_m) + h_\mu\left(T, \xi_n \,\middle|\, (\pi^{-1}\eta_m)_{-\infty}^{\infty}\right)$$

- $h_\mu(T, \pi^{-1}\eta_m) \underset{\pi_*\mu = \nu}{=} h_\nu(S, \eta_m)$
- $\pi^{-1}\eta_m \nearrow \mathcal{A}$

$$\xrightarrow{m,n\to\infty} \boxed{h_\nu(S) + h_\mu(T \mid \mathcal{A})}$$

$$h_\mu(T) \leftarrow h_\mu(T, \xi_n) \le h_\mu(T, \xi_n \vee \pi^{-1}\eta_m)$$

$$\le h_\mu(T)$$

Thm (Convex Combinations)

Sps $\mu = \int_Y \mu_y \, d\nu(y)$

$Y \ni y \mapsto \mu_y \in \mathrm{Prob}(X)$ T-invariant

(Y, ν) prob space

(For example Ergodic Decomp of μ)

Then $h_\mu(T) = \int_Y h_{\mu_y}(T) \, d\nu(y)$

<u>Proof</u>: $(X \times Y, \rho)$ $\quad \rho = \int_Y \mu_y \otimes \delta_y \, d\nu(y)$

(X, T), (Y, id_Y) factors $(X \times Y, \hat{T})$ $\left\{ \begin{array}{l} \pi_{X*}\rho = \mu \\ \pi_{Y*}\rho = \nu \end{array} \right.$, $\rho \circlearrowleft \hat{T}$

$\hat{T} = T \times id_Y$

Last week: $\rho_{(x,y)}^{\mathcal{I}_X \times \mathcal{B}_Y} = \mu_y \otimes \delta_y$

trivial σ-alg

Apply Abramov-Rokhlin twice:

$$h_\rho(\hat{T}) = \underbrace{h_\nu(id_Y)}_{=0} + h_\rho(\hat{T} \mid \mathcal{I}_X \times \mathcal{B}_Y)$$

$$= h_\mu(T) + \underline{h_\rho(\hat{T} \mid \mathcal{B}_X \times \mathcal{I}_Y)}$$

$$= 0$$

$$\xi_n \times \eta_n \nearrow \mathcal{B}_X \times \mathcal{B}_Y$$

$$h_\rho(\hat{T} \mid \mathcal{B}_X \times \mathcal{I}_Y) = \lim_n \underbrace{h_\rho(\hat{T}, \xi_n \times \eta_n \mid \mathcal{B}_X \times \mathcal{I}_Y)}_{\stackrel{!}{=} 0}$$

$$(h_\mu(T, \xi \vee \eta) \le h_\mu(T, \xi) + h_\mu(T, \eta))$$

$$\le h_\rho(\hat{T}, \xi_n \times \{\phi, Y\} \mid \mathcal{B}_X \times \mathcal{I}_Y) = 0$$

$$+ h_\rho(\hat{T}, \{\phi, X\} \times \eta_n \mid \mathcal{B}_X \times \mathcal{I}_Y)$$

zero entropy prop

$$h_\rho(\qquad) = H_\mu(\xi_n \mid (\xi_n)_1^\infty \vee \mathcal{B}_X)$$

$$= H_\mu(\xi_n \mid \mathcal{B}_X)$$

$$= 0 \quad \xi_n \subset \mathcal{B}_X$$

$$= h_\nu(\mathrm{id}_Y, \eta_n)$$

$$= H_\nu(\eta_n \mid \eta_n) = 0$$

$$\Rightarrow h_\mu(T) = h_\rho(\hat{T} \mid \mathcal{I}_X \times \mathcal{B}_Y)$$

$$\xi \times \eta \quad (\xi_n \times \eta_n \nearrow \mathcal{B}_X \otimes \mathcal{B}_Y)$$

$$h_\rho(\hat{T}, \xi\times\eta \mid \mathcal{I}_X \times \mathcal{B}_Y)$$

$$= \lim \frac{1}{n} H_\rho\left((\xi\times\eta)_0^n \mid \mathcal{I}_X\times\mathcal{B}_Y\right)$$

$$= \lim \frac{1}{n} \int H_{\mu_y\otimes\delta_y}(\xi_0^n\times\eta)\, d\nu(y)$$

$$= \lim \frac{1}{n} \int H_{\mu_y}(\xi_0^n)\, d\nu(y)$$

Dominated convergence

$$= \int h_{\mu_y}(T,\xi)\, d\nu(y) \to \int h_{\mu_y}(T)\, d\nu(y)$$

$\xi = \xi_n \nearrow \mathcal{B}_X$ by KS & monotone conv.

$\blacksquare$

$$\lim \frac{1}{n+1} H_{\mu_y}(\xi_0^n) \overset{\text{def}}{=} h_{\mu_y}(T,\xi)$$

$$H_{\mu_y}(\xi_0^n) \le (n+1) H_{\mu_y}(\xi)$$

$$H(\xi\vee\eta) \le H(\xi)+H(\eta)$$

$$\Rightarrow \frac{1}{n+1} H_{\mu_y}(\xi_0^n) \le H_{\mu_y}(\xi)$$

$$\xi_0^n = \xi\vee\dots\vee T^{-n}\xi$$

$$H_{\mu_y}(\xi) \le \log \# \xi$$

$\blacksquare$

<u>Definition</u>: (Pinsker algebra)

$$\mathcal{P}(T) = \{ B \in \mathcal{B}_X : h_\mu(T, \{B, B^c\}) = 0 \}$$

<u>Note</u>: By zero entropy prop

$\xi = \{B, B^c\}$ $\quad h_\mu(T, \xi) = H_\mu(\xi \mid \xi_1^\infty) = 0$

$$\Leftrightarrow \xi \subset \xi_1^\infty$$

$$\mathcal{P}(T) = \{ B \in \mathcal{B}_X : B \in \{B, B^c\}_1^\infty \}$$

<u>Prop</u>: ① $\mathcal{P}(T)$ T-inv σ-algebra

② Hence "Pinsker factor"

$$(X, T, \mathcal{B}) \longrightarrow (X, T, \mathcal{P}(T))$$

has zero entropy, and is the maximal factor w/ that property

and $\quad h_\mu(T) = h_\mu(T \mid \mathcal{P}(T))$

Proof: $\{\xi_i\}_{i\geq 0}$ two element partitions $\xi_i \subset \mathcal{P}(T)$

Need to show $\forall\, \xi \subset \bigvee_{i\geq 0} \xi_i$ we have $\xi \subset \mathcal{P}(T)$ where $\xi = \{B, B^c\}$

$$h_\mu(T,\xi) \leq h_\mu(T, \bigvee_{i\leq n} \xi_i) + H_\mu(\xi \mid \bigvee_{i\leq n} \xi_i)$$ ("continuity bound")

$$h_\mu(T, \bigvee_{i\leq n} \xi_i) = 0$$

$$= H_\mu(\xi \mid \bigvee_{i\leq n} \xi_i)$$ ("continuity")

$$\to H_\mu(\xi \mid \bigvee_{i\geq 0} \xi_i) = 0$$

Zero entropy prop

<u>T-invariant:</u>

- <u>$T^{-1}\mathcal{P} \subset \mathcal{P}$</u> $0 = h_\mu(T,\xi) = h_\mu(T, T^{-1}\xi)$
- $\mathcal{P} \subset T^{-1}\mathcal{P}$

$\xi \subset \mathcal{P}$ $\quad \xi \subset \xi_1^\infty = T^{-1}\xi_0^\infty \subset T^{-1}\mathcal{P}$

$\xi \subset \mathcal{P}$ $\quad T^{-1}\xi \subset \mathcal{P} \Rightarrow \xi_0^\infty \subset \mathcal{P}$

$$h_\mu(T) = h_\mu(T|_{\mathcal{P}(T)}) + h_\mu(T\,|\,\mathcal{P}(T))$$

$$\|$$

$0 = h_\mu(T|_A)$

Sps $A \subset \mathcal{B}_X$

$TA = A$

$$\sup\{h_\mu(T\,|\,\xi) : \xi \text{ finite}, \ \xi \subset \mathcal{P}(T)\}$$

$\xi \subset A \subset \mathcal{P}$

$$\|\| \quad 0$$

$$= h_\mu(T\,|\,\mathcal{P}(T))$$

∎

Definition: (Tail algebra)

$\mathcal{A} \subset \mathcal{B}_X$ σ-alg

$$\bigcap_{n\geq 0} \mathcal{A}_n^\infty = \bigcap_n \bigvee_{j\geq n} T^{-j}\mathcal{A}$$

Prop: $$\mathcal{P}(T) = \bigvee_{\xi : H_\mu(\xi)<\infty} \bigcap_n \xi_n^\infty$$

Proof: Sps $\eta \subset \bigcap_n \xi_n^\infty$ need to show $h_\mu(T,\eta)=0$

[J]

$\boxed{\eta_{-\infty}^\infty \subset \xi_1^\infty}$ same $\eta \subset \xi_m^\infty$ $\forall m$

$$\eta_{-n}^n \underset{\forall n}{\subset} \xi_1^\infty \Leftarrow \eta_{-n}^n \subset \xi_{m-n}^\infty \Leftarrow \eta_0^{2n} \subset \xi_m^\infty \quad \begin{matrix}\forall m \\ \forall n\end{matrix}$$

$$h_\mu(T,\xi) = H_\mu(\xi \mid \xi_1^\infty) \underset{\eta \subset \xi_1^\infty}{=} H_\mu[\xi\vee\eta \mid (\xi\vee\eta)_1^\infty]$$

$$= h_\mu(T, \xi\vee\eta)$$

$$= \underbrace{h_\mu(T,\eta)}_{=0} + \underbrace{h_\mu(T, \xi \mid \eta_{-\infty}^{\infty})}_{= h_\mu(T,\xi)}$$

[c] $\eta \subset \underline{\eta_1^\infty} \subset \eta_n^\infty \quad \forall n$

$\Rightarrow \eta \subset \bigcap_n \eta_n^\infty$ ∎

<u>Def</u>: T is CPE (completely positive entropy) if $\forall\, \xi$ non-trivial partitions ($\Leftrightarrow H_\mu(\xi) > 0$) we have $h_\mu(T,\xi) > 0$

$(\Leftrightarrow \Theta(T) = \mathcal{I}_X)$

<u>Def</u>: $T \circlearrowright K$ (Kolmogorov)

$\exists\, \mathcal{K} \subset \mathcal{B}_X$ σ-alg

1. $T^{-1}\mathcal{K} \subset \mathcal{K}$
2. $\mathcal{K}_{-\infty}^{0} = \mathcal{B}_X$
3. $\bigcap_n \mathcal{K}_0^n = \mathcal{I}_X$

Example: twosided Bernoulli shift

<u>Thm</u>: (Rokhlin-Sinai)

$K \Leftrightarrow CPE$

<u>Rmk</u>: $CPE \Rightarrow T$ is mixing of all orders

Question: CPE $\Rightarrow$ Bernoullishift?

'73 Ornstein: no

counterexample

Another candidate

(T, T^{-1})-process

(X, T) two sided Bernoulli

$x = (\dots x_{-1} x_0 x_1 \dots)$ over $\{1, -1\}$

$S : X \times X \to X \times X$

$(x, y) \mapsto (Tx, T^{x_0} y)$

82' Kalikow Yes: CPE

but not

Bernoulli.

Question ('60 Pinsker)

(T, X) $\exists ?$ (T_1, X_1), (T_2, X_2)

$X = X_1 \times X_2$

$T = T_1 \times T_2$

where T_1 CPE

T_2 zero entropy

'73 Ornstein : No

'77 Thouvenot : "weak Pinsker conjecture"

$\forall \varepsilon > 0 \ \exists T_1, T_2 \quad T = T_1 \times T_2$

T_1 Bernoulli

T_2 entropy $\leq \varepsilon$

2017 Austin : Yes, it's true

K $\Rightarrow$ multiple mixing $\forall k$

(Rokhlin notes '67)

want to show $f_0, f_1, \dots, f_k \in L^2(X)$

$f_0 \perp 1$

$\perp L^2(\Theta(T))$

$$\int \prod_{i=0..k} f_i \circ T^{in} \, d\mu \to 0$$

$$\|$$

$$\langle f_0, g_n \rangle \qquad g_n = f_1 \circ T^n \cdots f_k \circ T^{kn}$$

$$\|$$

$$\langle f_0 \quad , g_n \quad \rangle$$

$\mathcal{K}^0_{-N} \nearrow \mathcal{B}_X$

Approximate f_i $\mathcal{K}^0_{-N}$ measurable

g_n it is $\mathcal{K}^{kn}_{-N+n}$ ← measurable

$\in L^2(\mathcal{K}^{kn}_{-N+n})$

$$= \langle \pi_{L^2(\mathcal{K}^{kn}_{-N+n})}(f_0 \quad), g_n \quad \rangle$$

$n \to \infty$ $\downarrow$ $\searrow$ Tail $= \mathcal{I}_X$

$$\pi_1(f_0) = 0$$

$$\leq \underbrace{\|\pi f_0\|}_{\downarrow\, 0} \, \underbrace{\|g_a\|}_{\leq C}$$