Lecture 7

$A : \Omega \to GL_d(\mathbb{R})$

$$
\begin{tikzcd}
\Theta \times A = \hat{\Theta} \arrow[loop left] & (\Omega \times \mathbb{P}^{d-1}, \mu) \arrow[r, "\pi_\Omega"] \arrow[d, "\hat{\pi} = \pi \times id"] & (\Omega, m) \arrow[d, "\pi"] & \Theta \arrow[l, bend right] & \mathcal{A} \subset \mathcal{B}_\Omega = \pi^{-1}\mathcal{B}_X \\
\hat{T} \arrow[loop left] & (X \times \mathbb{P}^{d-1}, \nu) \arrow[r, "\pi_X"] & (X, p) & T \arrow[l, bend right] &
\end{tikzcd}
$$

$\hat{T}(x,v) = (Tx, A_x v)$

- $\omega \mapsto \log \|A_\omega\| \|A_\omega^{-1}\| \in L^1(m)$
- $\Theta^{-1}\mathcal{A} \subset \mathcal{A}$
- A_ω is $\mathcal{A}$ meas.

$\mu = \int \delta_\omega \otimes \mu_\omega \, dm(\omega)$

$\nu = \int \delta_x \otimes \nu_x \, dp(x)$

— X —

$\mathcal{E} \subset \mathcal{B}_\Omega$ $\quad \mu_\omega^{\mathcal{E}} = \pi_{\mathbb{P}} * \mu_{\omega, \cancel{x}}^{\mathcal{E} \times \mathcal{I}}$ $\quad \mathcal{I}$ = trivial σ-alg on $\mathbb{P} = \mathbb{P}^{d-1}$

- measure on $\mathbb{P}$
- $\mathcal{E}$ measurable

E.g.: $\mu_\omega = \mu_\omega^{\mathcal{B}_\Omega}$

- $\mu_\omega^{\mathcal{E}}(g) = \mathbb{E}_m[\omega \mapsto \mu_\omega(g) \mid \mathcal{E}](\omega)$

$g : \mathbb{P} \to \mathbb{R}$ $\quad = \mathbb{E}_m[\mu_\bullet(g) \mid \mathcal{E}](\omega)$

$\forall A \in \mathcal{E}$, $\int_A \mu_\omega^{\mathcal{E}}(g) \, dm = \int_{A \times \mathbb{P}} \mu_\omega^{\mathcal{E} \times \mathcal{I}}(\mathbb{1}_\Omega \otimes g) \, d\mu$

$$= \int_{A \times \mathbb{P}} \mathbb{1}_\Omega \, g \, d\mu = \int_A \mu_\omega(g) \, dm(\omega)$$

- $\mu_\omega^{\mathcal{A}} = \nu_{\pi(\omega)}$

Need:

$$\nu_{\pi(\omega)}(g) = \mathbb{E}[\mu_\cdot(g) \mid \mathcal{A}](\omega)$$

$$A \in \mathcal{A} = \pi^{-1}\mathcal{B}_X \qquad A = \pi^{-1}(\pi(A))$$

$$\int_A \nu_{\pi(\omega)}(g) \, dm(\omega) = \int \mathbb{1}_{\pi(A)} \circ \pi \, \nu_{\pi(\omega)}(g) \, dm$$

$$= \int_{\pi(A)} \nu_x(g) \, dp = \nu(\mathbb{1}_{\pi(A)} \otimes g)$$

$$= \hat{\pi}_* \mu(\mathbb{1}_{\pi(A)} \otimes g) = \mu(\mathbb{1}_A \otimes g)$$

$$= \int_A \mu_\omega(g) \, dm(\omega)$$

Lemma: $\mathcal{A}_{-\infty} = \bigvee_{n \geq 0} \Theta^{-n}\mathcal{A}$

$$\mu_\omega^{\mathcal{A}_{-\infty}} = \lim A^n_{\pi(\Theta^{-n}\omega)*} \nu_{\pi(\Theta^{-n}\omega)}$$

Proof:

$$\mu_\omega^{\mathcal{A}_{-\infty}}(g) \underset{\substack{\uparrow \\ \text{Martingale} \\ \text{Convergence}}}{=} \lim \mathbb{E}[\mu_\omega(g) \mid \Theta^{-n}\mathcal{A}]$$

$$\pi_{\mathbb{P}*} \mu_\omega^{\hat{\Theta}^{-n}(\mathcal{A} \times \mathcal{I})}(g) = \mu_\omega^{\hat{\Theta}^{-n}\mathcal{A} \times \mathcal{I}}(\mathbb{1}_\Omega \otimes g)$$

$$\underset{\text{Lemma 2 Lect 4}}{=} \hat{\Theta}^n_{x*}\, \mu^{A\times\Sigma}_{\Theta^{-n}_\omega} (\mathbb{1}_\Omega \otimes g)$$

$$= \hat{T}^n_{*}\, \nu^{\mathcal{B}_X\times\Sigma}_{\pi(\Theta^{-n}_\omega)} (\mathbb{1}_X \otimes g)$$

$$= A^n_{\pi(\Theta^{-n}_\omega)*}\, \nu_{\pi(\Theta^{-n}_\omega)}(g) \qquad \blacksquare$$

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$$\gamma^{\pm} = \lim_{n\to\infty} \frac{1}{\pm n} \mathbb{E}_m \left[\log \| (A^n_\omega)^{\pm 1} \| \right]$$

Thm: • $\gamma^+ = \gamma^-$

• $\omega \mapsto \mu_\omega$ $A_{-\infty}$-meas

$\Longrightarrow$ $\omega \mapsto \mu_\omega$ A-measurable

Thm 2: $J(x,v) = \dfrac{d\, A^{-1}_x * \nu_{Tx}}{d\nu_x}(v)$

$$a_\nu(\hat{T}) = \int -\log J \, d\nu$$

$$a_\nu \leq (d-1)(\gamma^+ - \gamma^-)$$

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Def: $\varepsilon > 0$ $\quad d_\varepsilon(x,v) = \dfrac{\log \nu_x(B(v,\varepsilon))}{\log \varepsilon}$

$\chi > 0$ $\quad \beta(\chi) = \sup\{ t > 0 \mid \exists \varepsilon_t\ \forall \varepsilon < \varepsilon_t :$

$\qquad \nu\big((x,v) : d_\varepsilon(x,v) > t\big) \geq 1 - \chi \}$

$\dim \nu = \lim_{x \to 0} \beta(x)$

<u>Prop</u>: Ass ν is $\hat{T}$-ergodic then

- $\dim \nu \leq d-1$
- $a_\nu \leq \dim \nu \; \mathbb{E}_m[\log \|A\| \|A^{-1}\|]$

<u>Proof of Thm2</u>

- $a_\nu \leq (d-1)\left(\mathbb{E}[\log\|A\|] + \mathbb{E}[\log\|A^{-1}\|]\right)$

$\Rightarrow$ True even w/o ergodicity (Fact)

Appy $\hat{T}^n$ $\forall n$

$a_\nu(\hat{T}^n) \leq (d-1)\left(\mathbb{E}[\log\|A^n\|] + \mathbb{E}[\log\|A^{-n}\|]\right)$

<u>Claim</u>: $a_\nu(\hat{T}^n) = n\, a_\nu(\hat{T})$

<u>Chain rule</u>: $\mu \ll \eta \ll \nu$

$\frac{d\mu}{d\eta}\frac{d\eta}{d\nu} = \frac{d\mu}{d\nu}$

$a_\nu < \infty \Rightarrow J > 0$ ae $\Rightarrow A_x^{-1}\nu_{Tx} \ll \nu_x$

$a_\nu = \lim_n \frac{a_\nu(\hat{T}^n)}{n} \leq (d-1)(\gamma^+ - \gamma^-)$ ∎

Proof that $\dim \nu \le d-1 = D$:

Follows from: ν on $\mathbb{P}^D$

then $\limsup_{\delta \to 0} \frac{\log \nu(B(v,\delta))}{\log \delta} =: g^*(v)$

$\le D$ $\quad$ ν-a.e. v

Proof: $E = \{ v : g^*(v) > D + \varepsilon \}$

want $\nu(E) = 0$

$E_k = \{ v : \nu(B(v, 2^{-k})) < 2^{D+\varepsilon} \, 2^{-k(D+\varepsilon)} \}$

Claim: $E \subset \bigcap_n \bigcup_{k \ge n} E_k$

$v \in E$ $\quad \exists \infty_{\text{many}} \, \delta_n$ s.t. $\quad \nu(B(v, \delta_n)) < \delta_n^{D+\varepsilon}$

$\forall n \, \exists k \quad 2^{-k} \le \delta_n \le 2 \cdot 2^{-k}$

2^{-k} ball $\subset \delta_n$-ball

size $\le 2^{D+\varepsilon} \, 2^{-k(D+\varepsilon)}$

$\Rightarrow v \in E_k$ $\square$

By Borel-Cantelli want $\nu(E_k)$ summable

Cover each E_k by $\ll 2^{D \cdot k}$ - many 2^{-k} balls

indep of k

follows from BCT

$\nu(E_k) \ll 2^{Dk} \, 2^{-k(D+\varepsilon)}$

$= 2^{-k\varepsilon}$ $\quad$ summable $\blacksquare$

Besicovitch - Covering - Thm

$\exists C = C(d)$ st $\forall A \subset \mathbb{P}^{d-1}$

any $A \ni v \mapsto r(v) \in \mathbb{R}_{>0}$

$$A \subset \bigcup_{v \in A} B(v, r(v))$$

$\exists$ countable subcover st each pt is covered $\leq C$ times

need the following Corollaries

Sps η, ν probs on $\mathbb{P}$

- $\lim_{\delta \to 0} \frac{\eta(B(v,\delta))}{\nu(B(v,\delta))} = \frac{d\eta}{d\nu}(v)$ ν-ae

- $J^*(v) = \sup_\delta \frac{\eta(B(v,\delta))}{\nu(B(v,\delta))} \geq 1$

$$\int \log J^* d\nu \leq C$$

Proof of 2nd bulletpt

Maximal equality $\nu(J^* > t) \leq C \frac{1}{t}$

Let $A = \{J^* > t\}$, for $v \in A$ pick $r(v)$ st

$\eta(B(v, r(v))) \geq t\, \nu(B(v, r(v)))$

$\exists$ subcover $\{B\}$

$\nu(A) \leq \sum \nu(B) \leq \frac{1}{t} \sum \eta(B) \leq \frac{C}{t}$ $\perp$

$$\int \log J^* \, d\nu = \int d\nu \int dt \, \mathbb{1}_{\{J^* \geq e^t\}}$$

$$\leq \int dt \frac{C}{e^t} = C \qquad \blacksquare$$

Def: $\phi(A) = \frac{1}{\|A\| \|A^{-1}\|}$; $\phi(x) = \phi(A_x)$

Lemma: $\forall \varepsilon > 0 \ \exists \delta_0(\varepsilon) \ \forall \delta < \delta_0(\varepsilon)$
$\forall A \in GL_d(\mathbb{R}) \quad \forall v \in \mathbb{P}^{d-1}$

$$A B(v, \delta) \supset B(Av, \delta \phi(A) e^{-\varepsilon})$$

⌈ use KAK to reduce to diagonal case ⌋

— ✱ —

$$J(x,v) = \frac{d A_x^{-1} \nu_{Tx}}{d \nu_x} \qquad J^*_{(x,v)} = \sup_\delta \frac{\nu_{Tx}(A_x B(v,\delta))}{\nu_x(B(v,\delta))}$$

- 2^nd bullet point $\rightsquigarrow \int \log J^* \, d\nu < \infty$

Prop: $\varepsilon > 0 \ \exists \ \delta_\varepsilon(x,v) > 0$ ν ae measurable

$\forall \delta < \delta_\varepsilon(x,v)$

$$(*) = \frac{\nu_{Tx}(B(A_x v, \delta \phi(A) e^{-\varepsilon}))}{\nu_x(B(v,\delta))} \leq e^{\varepsilon} J(x,v) \mathbb{1}_{\{J>0\}} + e^{-\frac{1}{\varepsilon \nu(J=0)}} \mathbb{1}_{\{J=0\}}$$

$\forall \delta < \delta_0(\varepsilon)$ (as in Lemma)

$$|x| \leq J^*(x,v)$$

Prop $\varepsilon > 0$ ν a.e. (x,v) $n >> 1$

$$-\frac{1}{n} \log \nu_{T^n x}\left(B(A^n v, r_n)\right) \geq \begin{cases} a_\nu - 4\varepsilon & \text{if } a_\nu < \infty \\ \frac{1}{\varepsilon} & a_\nu = \infty \end{cases}$$

$$r_n = r_n(x) = \delta_0(\varepsilon)\, e^{-n\varepsilon} \prod_{j<n} \phi\left(A_{T^j x}\right)$$

Proof: Telescope & Erg Thm:

$$\nu_{T^n x}\left(B(A^n_x v, r_n)\right) = \nu_x\left(B(v, \delta_0(\varepsilon))\right) \prod q_j(x,v)$$

$$q_j(x,v) = \frac{\nu_{T^{j+1}x}\left(B(A^{j+1}_x v, r_{j+1})\right)}{\nu_{T^j x}\left(B(A^j_x v, r_j)\right)}$$

$$r_{j+1} = r_j\, \phi(A_{T^j x})\, e^{-\varepsilon}$$

r_j monotone N large st $j \geq N$ $r_j \leq \delta \leq \delta_0(\varepsilon)$

(δ to be chosen)

$$q_j(x,v) \leq \begin{cases} e^{\varepsilon} J \circ \hat{T}^j(x,v) & \delta < \delta_\varepsilon \circ T^j \quad J > 0 \\ e^{-\frac{1}{\varepsilon \nu(J=0)}} & \delta < \delta_\varepsilon \circ T^j \quad J = 0 \\ J^* \circ \hat{T}^j & \text{else} \end{cases}$$

so,

$$-\frac{1}{n} \log \prod_{j<n} q_j \geq -\frac{1}{n} \sum_{j<N} \log q_j - \varepsilon + \frac{1}{n} \sum_{j \in [N,n]} -\log g \circ \hat{T}^j$$

where

$$g(x,v) = \begin{cases} \zeta & \delta < \delta_\varepsilon(x,v) \quad \zeta > 0 \\ e^{-\frac{1}{\varepsilon\nu(\zeta=0)}} & \zeta = 0 \\ \zeta^* & \delta > \delta_\varepsilon(x,v) \end{cases}$$

Apply Ergodic Thm

$$\geq -2\varepsilon - \int \log g$$

$$\| $$

$$\underbrace{\int_{\substack{\delta<\delta_\varepsilon \\ \zeta>0}} -\log \zeta}_{\geq\, a_\nu - \varepsilon} + \underbrace{\int_{\substack{\delta<\delta_\varepsilon \\ \zeta=0}} \frac{1}{\varepsilon\nu(\zeta=0)}}_{\geq\, \frac{1}{\varepsilon} - \varepsilon} - \underbrace{\int_{\delta>\delta_\varepsilon} \log \zeta^*}_{\geq\, -\varepsilon}$$

$a_\nu < \infty$ $\qquad$ $a_\nu = \infty$

$\square$

Proof of $a_\nu \leq \overline{\dim}\,\nu \; \mathbb{E}[-\log\phi]$

Apply erg Thm to $\log\phi(A_x)$ $\quad \hat{T}$

$\forall\, \varepsilon, \chi > 0 \quad \forall n >> 1$

$$D_1 = \{(x,v) : \prod_{j<n} \phi(A_{T^j x}) \geq e^{n\,\mathbb{E}[\log\phi] - \varepsilon n}\}$$

$\nu(D_1) > 1-\chi$

By prev prop

$\leq r_n$ on D_1

$$D_\varepsilon = \left\{ (x,v) \ \ -\frac{1}{n}\log \nu_{T^n x}\left(B\left(A_x^n v,\ \delta_0(\varepsilon)\, e^{-2n\varepsilon + n\mathbb{E}[\log\phi]}\right)\right) \geq \begin{cases} a_\nu - 4\varepsilon & a_\nu < \infty \\ \frac{1}{\varepsilon} & a_\nu = \infty \end{cases} \right\}$$

$$\nu(D_\varepsilon) \geq 1 - 2\gamma$$

$$\beta(r) \geq \begin{cases} a_\nu - 4\varepsilon \,/\, 2\varepsilon + \mathbb{E}[-\log\phi] \\ \frac{1}{\varepsilon} \,/\, \quad -\|- \end{cases}$$

$$\begin{array}{l} \dim\nu = \lim_{r\to 0} \beta(r) \\ \quad \wedge| \\ \ d-1 \end{array}$$

$$\dim\nu \geq a_\nu \,/\, \mathbb{E}[-\log\phi] \qquad \boxtimes$$