Ledrappier's paper on "Positivity of Exponents"

<u>Setup</u>:

$(\Omega, \mathcal{B}, m)$ prob space $\Theta: \Omega \to \Omega$ invertible, $\Theta_* m = m$

$A = \Omega \to GL_d(\mathbb{R})$ $\int_\Omega \log(\max \|A\|, \|A^{-1}\|)\, dm < \infty$

"skew product"

$\hat{\Theta}: \hat{\Omega} \to \hat{\Omega}$ $\hat{\Omega} = \Omega \times \mathbb{P}(\mathbb{R}^d)$

$(\omega, v) \mapsto (\Theta\omega, A_\omega v)$

$A^n_\omega = A_{\Theta^{n-1}\omega} \cdots A_{\Theta\omega} A_\omega$

Lyapunov exponents: $\int \log \|A^n_\omega\| dm$ subadditive (by $\|A^{n+m}_\omega\| \leq \|A^m_{\Theta^n\omega}\| \|A^n_\omega\|$)

By <u>Fekete</u>: $\lim_{n \to \infty} \frac{1}{n} \int \log \|A^n_\omega\|\, dm = \gamma^+$

$\lim_{n \to \infty} \frac{1}{-n} \int \log \|(A^n_\omega)^{-1}\|\, dm = \gamma^-$

<u>RMK</u>: $\|B\| \|B^{-1}\| \geq 1 \Rightarrow \gamma^+ \geq \gamma^-$

$\log \|A^n\| \|(A^n)^{-1}\| \geq 0$

$= \log \|A^n\| + \log \|A^{n^{-1}}\| \geq 0$

$\frac{1}{n}\int$ $\downarrow$ γ^+ $\downarrow$ $-\gamma^-$

μ prob on $\hat{\Omega} = \Omega \times \mathbb{P}$ μ is $\hat{\Theta}$-inv, $\pi_\Omega * \mu = m$ "The marginal of μ on Ω is m"

Disintegrate wrt $\pi_\Omega^{-1}(\mathcal{B}) = \mathcal{B} \times \{\phi, \mathbb{P}\}$

$$\mu = \int \mu^{\pi_\Omega^{-1}\mathcal{B}}_{\omega, v}\, d\mu(\omega, v)$$

Note: Can rewrite $\mu^{\pi_\Omega^{-1}\mathcal{B}}_{\omega, v} = \delta_\omega \times \mu_\omega$ μ_ω prob on $\mathbb{P}$

$\mathbb{E}[fg|\mathcal{A}] = f\,\mathbb{E}[g|\mathcal{A}]$ if f $\mathcal{A}$-measurable

Apply $f(\omega)g(v)$

Also $\mu^{\pi^{-1}\mathcal{B}}_{\omega,v} = \mu^{\pi^{-1}\mathcal{B}}_{\omega}$ $\rightsquigarrow$ $\boxed{\mu = \int \delta_\omega \otimes \mu_\omega \, dm(\omega)}$

<u>Def</u>: $\mathcal{A} \subset \mathcal{B}$ decreasing wrt Θ if $\Theta^{-1}\mathcal{A} \subset \mathcal{A}$

<u>Not</u>: $\sigma(A_\omega)$ = smallest σ-algebra for which $\omega \mapsto A_\omega$ is measurable

$$\mathcal{A}_{-\infty} = \sigma\Big(\bigcup_{n\geq 0} \Theta^n \mathcal{A}\Big) \Big(\text{"} = \mathcal{A}^{0}_{-\infty} \text{" in notation of prev lectures}\Big)$$

Theorem (Ledrappier '86)

Assume $\mathcal{A} \subset \mathcal{B}$ Θ-decreasing, $\sigma(A) \subset \mathcal{A}$

<u>Suppose</u>: i) $\gamma^+ = \gamma^-$

ii) $\omega \mapsto \mu_\omega$ $\mathcal{A}_{-\infty}$-measurable

then $\omega \mapsto \mu_\omega$ is $\mathcal{A}$-measurable

<u>Example</u> (special case of Furstenberg '63)

$\Omega = \{a,b\}^{\mathbb{Z}}$ Θ shift $m = (p_1\delta_a + p_2\delta_b)^{\mathbb{Z}}$ $p_1+p_2=1$

$a = \begin{bmatrix} 2 & \\ & \frac{1}{2} \end{bmatrix}$, b = irrat rotation $\omega = (\dots, \omega_{-1}, \omega_0, \omega_1 \dots)$

$A_\omega = \omega_0$ then $\gamma^+ > \gamma^-$, $(\Leftrightarrow \gamma^+ > 0)$

Corollary: $(\Omega, \mathcal{B}, m)$, A_ω as above

<u>Suppose</u> i) no measure on $\mathbb{P}(\mathbb{R}^d)$ A_ω-invariant for m-a.e. ω

ii) $\sigma(\{A_{\Theta^n\omega}\}\ n\geq 0\}) \cap \sigma(\{A_{\Theta^n\omega}\}\ n<0) = \text{trivial}$

$\Longrightarrow \gamma^+ > \gamma^-$

Deduce example from cor:

i) a-inv meas are atomic on $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

b-inv meas is Lebesgue

ii) $\sigma(\{A_{\Theta^n \omega}\}\, n \geq 0\} = \sigma($right-infinite sequences $(\omega_0, \omega_1, \omega_2, \ldots))$

$A_{\Theta^n \omega} = \omega_n$ $\quad \sigma(\{A_{\Theta^n \omega}\}\, n < \infty) = \sigma$-alg left-infinite sequences

$\rightsquigarrow$ trivial intersection

Proof of Cor:

$\mathcal{C} = \sigma(A_\omega)$, assume that $\mathcal{C}$ is generating wrt Θ, $\mathcal{C}_{-\infty}^{\infty} = \mathcal{B}$

(If not, $\mu = \int \mu_\omega^{\pi^{-1}\mathcal{B}}\, dm(\omega)$ by $\mu' = \int \mu_\omega^{\pi^{-1}\mathcal{C}_{-\infty}^{\infty}}\, dm(\omega)$.)

$\rightsquigarrow$ $\omega \mapsto \mu_\omega$ is $\mathcal{C}_{-\infty}^{\infty} = \mathcal{B}$-measurable

$\mathcal{A} = \mathcal{C}_0^\infty$ (decreasing) $\mathcal{A}_{-\infty} = \mathcal{C}_{-\infty}^{\infty} = \mathcal{B}$

$= \sigma(\bigcup_{n \geq 0} \Theta^{-n} \mathcal{C})$

Assume by contradiction that $\gamma^+ = \gamma^-$.

Apply Thm: $\omega \mapsto \mu_\omega$ is $\mathcal{A} = \mathcal{C}_0^\infty$ - measurable

Repeat the argument applied $(\Omega, \mathcal{B}, m)$, Θ^{-1}, $A^{-1} \circ \Theta^{-1}$)

$\omega \mapsto A^{-1}_{\Theta^{-1}\omega}$

$\Theta^{-1}\omega \sim \omega \sim \Theta\omega$

$v \mapsto A_\omega v$

$v \in$ Fiber at ω

$A_\omega v \in$ Fiber at $\Theta\omega$

A_ω^{-1} lives $\Theta\omega$

$A^{-1}_{\Theta^{-1}\omega}$ action at fiber over ω

Equal Lyap for A_ω imply equal exp's for $A^{-1}_{\Theta^{-1}\omega}$

$\tilde{\mathcal{A}} = \mathcal{C}_{-\infty}^{1}$ $\quad \tilde{\mathcal{A}}_{-\infty} = \mathcal{C}_{-\infty}^{\infty}$

Thm $\rightsquigarrow$ μ_ω is also $\mathcal{C}^1_{-\infty}$-measurable (wrt $\hat{\Theta}^{-1}$) $\omega \mapsto \mu_\omega$ same disintegration

$\rightsquigarrow$ μ_ω $\mathcal{C}_0^\infty \cap \mathcal{C}^1_{-\infty}$-measurable

$\overset{\text{ass. ii)}}{\underset{\text{trivial}}{\|}}$

$\Rightarrow \mu_\omega = \nu$ cnst ν prob $\mathbb{P}(\mathbb{R}^{d-1})$

<u>Lemma:</u> μ is $\hat{\Theta}$-invariant $\Rightarrow \mu_{\Theta\omega} = A_\omega * \mu_\omega$ for m-ae ω

Assuming the Lemma:

$$\nu = \mu_{\Theta\omega} = A_\omega * \mu_\omega = A_\omega * \nu \quad \text{m-ae } \omega$$

$\Rightarrow \nu$ is A_ω-inv ae ω $\lightning$ i) $\blacksquare$

<u>Proof of Lemma.</u>

Recall (Lemma 2, Lecture 4) $\hat{\Theta}: \hat{\Omega} \to \hat{\Omega}$, $\mathcal{A}$ sub σ-alg of $\mathcal{B}(\hat{\Omega})$

$$\hat{\Theta}_* \mu^{\hat{\Theta}^{-1}\mathcal{A}}_{(\omega,v)} = \mu^{\mathcal{A}}_{\hat{\Theta}(\omega,v)}$$

$\mathcal{A} = \mathcal{B} \times \mathcal{I}$ $\quad \hat{\Theta}^{-1}\mathcal{A} = \mathcal{A}$ (RMK: $\hat{\Theta}$-invertible!)

test fct $\hat{\Omega}$ $\quad f(\omega) g(v)$

$$\delta_\omega(f \circ \Theta)\, A_\omega * \mu_\omega(g) = \delta_{\Theta(\omega)}(f)\, \mu_{\Theta\omega}(g)$$

$f = \mathbb{1}_\Omega$

$$A_\omega * \mu_\omega = \mu_{\Theta\omega} \quad \blacksquare$$

$(X, \mathcal{B}_X, p)$ $\quad T: X \to X$ $\quad T_* p = p$ $\quad A: X \to GL_d(\mathbb{R})$

ν on $X \times \mathbb{P}(\mathbb{R}^d)$ $\quad \Pi_{X*} \nu = p$ $\quad \hat{T}_* \nu = \nu$ $\quad \hat{T}(x, v) = (Tx, A_x v)$

Disintegrate $\nu = \int \delta_x \otimes \nu_x \, dp(x)$

Let $J(x, \cdot)$ be the Radon-Nikodym derivative of $A_x^{-1}{}_* \nu_{Tx}$ wrt ν_x

$$J(x, v) = \frac{d A_x^{-1}{}_* \nu_{Tx}}{d \nu_x}(v)$$

and we have the Lebesgue decomposition

$$\int f(A_x^{-1} v) \, d\nu_{Tx}(v) = \int f(v) \, J(x, v) \, d\nu_x + \int f(v) \, d\xi_x(v)$$

where $\xi_x \perp \nu_x$ "singular"

Definition: (Fiberwise entropy)

$$a_\nu(\hat{T}) = \int -\log J(x, v) \, d\nu(x, v)$$

Proposition: $a_\nu \geq 0$, and "=" iff for p a.e. x $\quad A_{x*} \nu_x = \nu_{Tx}$

Proof: Decompose the integral $\{J > 0\}$ and $\{J = 0\}$

$$\int_{J>0} -\log J \frac{d\nu}{\nu(J>0)} \geq -\log \frac{\nu(J)}{\nu(J>0)} \stackrel{!}{=} 0$$

because

$$\nu(J) = \int \nu_x(J(x, \cdot)) \, dp = \int A_x^{-1}{}_* \nu_{Tx}(1) - \xi_x(1) \, dp$$

$= 1 - \nu(J=0) = \nu(J>0)$

strict convexity of $-\log \Rightarrow$ ("=" $\Leftrightarrow J \equiv 1$

$\Leftrightarrow \varphi_x \equiv 0$

and $A_x^{-1}{}_* \nu_{Tx} = \nu_x$

p a.e. x ∎)

Thm: $a_\nu \leq (d-1)(\gamma^+ - \gamma^-)$

Proof of 1st Thm given 2nd Thm

$\Theta : \Omega \to \Omega \quad \Theta_* m = m \quad \mathcal{A} \subset \mathcal{B} \quad \Theta^{-1}\mathcal{A} \subset \mathcal{A}$

Fact: $\exists (X, \mathcal{B}_X, p) \quad T : X \to X, \quad \pi : \Omega \to X$
(Ch5, Ch6 EW) st

$$\begin{tikzcd} \Omega \arrow[r, "\Theta"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\times"] & \Omega \arrow[d, "\pi"] \\ X \arrow[r, "T"'] & X \end{tikzcd} \qquad \pi^{-1}(\mathcal{B}_X) = \mathcal{A}$$

$\pi^{-1}(\pi(\omega)) = [\omega]_{\mathcal{A}} \qquad (\omega \overset{\pi}{\mapsto} \mu_\omega^{\mathcal{A}})$

(Thm by Rokhlin)

Recall: $\sigma(A) \subset \mathcal{A}$

$\Rightarrow A$ also defined on X

$$\begin{tikzcd} \hat{\Omega} \arrow[r, "\hat{\Theta}"] \arrow[d] \arrow[dr, phantom, "\times"] & \hat{\Omega} \arrow[d] \\ \hat{X} \arrow[r, "\hat{T}"'] & \hat{X} \end{tikzcd} \qquad \mu \quad \pi_{\Omega *} \mu = m$$

$\nu = \hat{\pi}_* \mu \qquad \pi_{X*} \nu = p$

Notation: $\mu_\omega^{\mathcal{A}_{-\infty}}$ = RHS of $\mu_{(\omega,v)}^{\pi_\Omega^{-1}\mathcal{A}_{-\infty}} = \mu_{(\omega,v)}^{\mathcal{A}_{-\infty}\times\mathcal{I}}$

Lemma: $\mu_\omega^{\mathcal{A}_{-\infty}} = \lim_{n\to\infty} A^n_{\Theta^{-n}\omega} * \nu_{\pi(\Theta^{-n}\omega)}$

γ^+, γ^- are same for Ω, and X

Ass: 1) $\gamma^+ = \gamma^- \overset{\text{Thm2}}{\Longrightarrow} a_\nu = 0 \overset{\text{Prop}}{\Longrightarrow} A_x * \nu_x = \nu_{Tx}$ pae x

(2) $\omega \mapsto \mu_\omega$ is $\mathcal{A}_{-\infty}$-measurable

$\mu_\omega = \mu_\omega^{\mathcal{A}_{-\infty}}$

$A_{\Theta^{-k}\omega} * \nu_{\pi(\Theta^{-k}\omega)} \overset{1)}{=} \nu_{T\pi(\Theta^{-k}\omega)} = \nu_{\pi(\Theta^{-k+1}\omega)}$

$\mu_\omega^{\mathcal{A}_{-\infty}} = \lim_n A^n_{\Theta^{-n}\omega} * \nu_{\pi(\Theta^{-n}\omega)}$

$= \nu_{\pi(\omega)}$

Recall $\pi^{-1}(\mathcal{B}_X) = \mathcal{A}$

$\Rightarrow \mu_\omega^{\mathcal{A}_{-\infty}}$ is $\mathcal{A}$-meas (b/c ν_x is $\mathcal{B}_X$ meas)

$\Rightarrow \mu_\omega$ is $\mathcal{A}$-measurable

∎

Next: Lemma, Thm 2.