Lecture x3 — Ref: ELW, Viana-Oliveira, Aaronson

Bernoulli shift: $S = \{1, \dots, p\}$

$X = S^{\mathbb{N}}$ cmpct top space

Metric $\underline{x}, \underline{y} \in X$ $\quad d(\underline{x}, \underline{y}) = \min_{x_n \neq y_n} \frac{1}{2^n}$ (complete metric)

cylinders are balls: $[x_0, \dots, x_n] = B_{<2^{-(n-1)}}(\underline{x})$ open

$\underline{x} \in X$

$\underline{x} = (x_0, x_1, \dots)$

Balls are also closed $\quad [x_0, \dots, x_n]^c = \bigcup_{y_i \neq x_i} [y_0, \dots, y_n]$

"clopen balls"

λ prob on S $\quad \lambda(i) = p_i \quad \sum p_i = 1$

"$\mu = \lambda^{\mathbb{N}}$" $\quad \mu([x_0, \dots, x_n]) = \prod_{i=0}^{n} \lambda(x_i)$

$\mu : \{\text{finite union of cylinders}\} \to \mathbb{R}$ additive

Def semiring $\mathcal{C} \subset \mathcal{B}(X)$ st $A, B \in \mathcal{C}$ $A \cap B \in \mathcal{C}$, $A \setminus B = \dot{\bigcup}_{\text{finite}} C_i$, $C_i \in \mathcal{C}$

Def: σ-subadditivity: $\mu : \mathcal{C} \to \mathbb{R}$

$C_0, C_1, \dots \in \mathcal{C}$ $\quad C_0 \subset \bigcup C_i$

$\Rightarrow \mu(C_0) \leq \sum \mu(C_i)$

Carathedory Extension Thm

$\mu : \mathcal{C} \to \mathbb{R}$ $\quad \mathcal{C}$ semiring, additive, σ-subadditive

then μ extends to a measure on $\sigma(\mathcal{C})$.

Note by Heine Borel: $B \subset \bigcup_{\text{countable}} C_i \Rightarrow \exists$ finite sub cover

$\Rightarrow$ σ-subadd. follows from additivity.

Entropy of $T : X \to X$ $(T\underline{x})_i = x_{i+1}$

$$\xi = \{ [i] : i \in S \}$$

$$\xi_0^n = \bigvee_{j=0}^{n} T^{-j}\xi = \{\text{cylinders of length } n+1\}$$

$\xi_0^n \nearrow \mathcal{B}(X)$ ξ one sided generator

$$h_\mu(T) \overset{\text{Kol-Sin}}{=} h_\mu(T,\xi) = \lim_{n\to\infty} H_\mu(\xi \mid \xi_1^n)$$

indept: $\xi \perp \xi_1^n$

$$= \lim_n H_\mu(\xi) = -\sum p_i \log p_i$$

(notation $h(\underline{p})$)

Question of von Neumann (30's?)

Are two Bernoulli shifts isomorphic?

$$\begin{tikzcd}
(X_1,\mu_1) \arrow[r, "T_1"] \arrow[d, "\psi \text{ bij}"'] & (X_1,\mu_1) \arrow[d] \\
(X_2,\mu_2) \arrow[r, "T_2"] & (X_2,\mu_2)
\end{tikzcd}$$

Lemma: Entropy is invariant under iso.

Proof: $h_{\mu_2}(T_2) \le h_{\mu_1}(T_1)$

b/c $h_\mu(T) = \sup_{\#\xi<\infty} h(T,\xi)$

Cor (Kolmogorov '57): If the state entropies $h(\underline{p})$ differ, then not

Ornstein '70: Bernoulli shifts are isomorphic if and only if same entropy.

Kuratowski Thm: X uncountable separable complete metric

$$(X, \mathcal{B}(X)) \xrightarrow[\text{meas. bij}]{\cong} \left(\{0,1\}^{\mathbb{N}}, \mathcal{B}(\{0,1\}^{\mathbb{N}})\right)$$

$$h_\mu(T,\xi) = \lim_{n\to\infty} H_\mu(\xi \mid \xi_1^n)$$

$$\overset{\text{Grad}}{=} H_\mu(\xi \mid \xi_1^\infty)$$

Def: $\mathcal{A}$ countably generated if $\exists\, \xi_k$

$$\sigma\Big(\bigvee_{k\le n} \xi_k\Big) \nearrow \mathcal{A}$$

Note: may assume
- $\xi_k < \xi_{k+1}$ ...
- ξ_k finite

Atom: $[x]_{\mathcal{A}} = \bigcap_{k=1}^{\infty} [x]_{\xi_k}$

$$H_\mu(\xi|\eta) = \int d\mu(x)\, I_\mu(\xi|\eta)(x)$$

$$I(\xi|\eta)(x) = -\log \mu_{[x]_\eta}([x]_\xi)$$

$$\text{"}\mu_x^{\mathcal{A}} = \lim_{k\to\infty} \mu_{[x]_{\xi_k}}\text{"}$$

Thm: (Existence of Condional Expect)

Sps $\mathcal{Q}_k \nearrow \mathcal{A}$ $\quad f \in L^1(X,\mu)$

For μ.a.e x (dep on f)

① $\mathbb{E}[f|\mathcal{A}](x) = \lim\limits_{\text{exists}} \mu_{[x]_{\mathcal{Q}_k}}(f)$

② $\mathbb{E}[f|\mathcal{A}]$ $\mathcal{A}$-measurable
[ie cnst on $\mathcal{A}$-atoms]

③ $\int_A \mathbb{E}[f|\mathcal{A}]\, d\mu = \int_A f\, d\mu$ $\quad \underline{A \in \mathcal{A}}$

Def

Proof: ① $\overline{f}(x) = \limsup \mathbb{E}_\mu[f|\mathcal{Q}_k](x)$

$\underline{f}(x) = \liminf$

$S_{\alpha,\beta} = \{x : \underline{f}(x) < \alpha < \beta < \overline{f}(x)\}$

Want: $\mu(S_{\alpha,\beta}) = 0$ $\quad \alpha,\beta \in \mathbb{Q}$

$x \in S_{\alpha,\beta}$ $\quad \exists \quad a_i^x < b_i^x < a_{i+1}^x < \cdots$

$\mathbb{E}[f|\mathcal{Q}_{a_i^x}] < \alpha$

$\beta < \mathbb{E}[f|\mathcal{Q}_{b_i^x}]$

Fix i: $\quad A_i = \bigcup\limits_x [x]_{\mathcal{Q}_{a_i^x}} = \bigsqcup\limits_{x_j} [x_j]_{\mathcal{Q}_{a_i^{x_j}}}$

w/c

$(q_k < q_{k+1})$

similarly B_i

since $a_i < b_i$ $\quad A_i \supset B_i \supset A_{i+1}$

<u>assume f pos</u> $\quad \bigcap A_i = \bigcap B_i = \tilde{S} \supset S_{\alpha, \beta}$

$$\int_{A_i} f = \sum_{x_j} \int_{[x_j]_{q_{a_i^{x_j}}}} f \, d\mu = \sum \mu([x_j]_{q_{a_i^{x_j}}}) \mathbb{E}[f \mid q_{a_i^{x_j}}] < \alpha A_i$$

$\vee\!/$

$$\int_{B_i} f > \beta \mu(B_i)$$

$$\alpha \mu(\tilde{S}) \geq \beta \mu(\tilde{S})$$

$\mu(\tilde{S}) =$ null set

(2) ✓ $\quad$ (3) use $\quad A \in \mathcal{A}$ $\quad B \in \sigma(q_k)$

$$A \overset{\varepsilon}{\approx} B$$

$k > n$ $\quad (\mu_{[x]_{q_k}})_{[x]_{q_n}} = \mu_{[x]_{q_n}}$

dominated con.

Rokhlin Disintegration thm

(regular cond. expect)

$\mathcal{A} \subset \mathcal{B}(X)$

1. $\exists$ ae x $\exists$ $\mu_x^{\mathcal{A}} \in P(X)$ st $\forall f \in L^1$
$\mu_x^{\mathcal{A}}(f) = \mathbb{E}[f|\mathcal{A}](x)$
2. $\forall B \in \mathcal{B}$ $x \mapsto \mu_x^{\mathcal{A}}(B)$ $\mathcal{A}$-measurable
3. $\mu(B \cap A) = \int_A \mu_x^{\mathcal{A}}(B)\, d\mu(x)$
$A \in \mathcal{A}$

$\rightsquigarrow$ " $\mu = \int \mu_x^{\mathcal{A}}\, d\mu(x)$ "

Proof: May assume $X = \{0,1\}^{\mathbb{N}}$

$\mathcal{C} \subset \mathcal{B}(X)$ $\quad \mathcal{C} = \{\text{cylinders}\}$

countable

$\mathcal{C} \ni C \quad C \mapsto \mathbb{E}[\mathbb{1}_C | \mathcal{A}](x) \quad (*)$

RHS $\exists$ μ ae x dep C

RHS Def on conull set $\forall C \in \mathcal{C}$

$\mathbb{E}[f|\mathcal{A}](x)$ linear (in f)

so $(*)$ additive

on $\mathcal{C}$

Cara. $\Rightarrow$ $(*)$ extends to a measure on $\sigma(\mathcal{C})$

$= \mathcal{B}(X)$

2, 3 Trick: Monotone Class thm

$\mathcal{E} = \{$ sets 2 true $\} \supset \mathcal{C}$

$\uparrow$ monotone class

$B_i \in \mathcal{E}$ $B_i \subset B_{i+1}$

$B = \bigcap B_i$ or $B_i \supset B_{i+1}$

■

Rmk: $\mathcal{A}$ does not need to be c.g.

$\boxed{\mathbb{E}}$

Def: $\mathcal{A}, \mathcal{C}$ c.g σ alg

$$I_\mu(\mathcal{C}|\mathcal{A})(x) = -\log \mu_x^{\mathcal{A}}([x]_{\mathcal{C}})$$

$$H_\mu(\mathcal{C}|\mathcal{A}) = \int -\log \mu_x^{\mathcal{A}}([x]_{\mathcal{C}})\, d\mu$$

Prop: ① $\mathcal{C}_1 \subset \mathcal{C}_2$ $\quad I(\mathcal{C}_1|\mathcal{A}) \leq I(\mathcal{C}_2|\mathcal{A})$

② $I_\mu(\mathcal{C}|\mathcal{A})$ measurable

③ $\mathcal{C}_n \nearrow \mathcal{C}$ $\quad I(\mathcal{C}_n|\mathcal{A}) \nearrow I(\mathcal{C}|\mathcal{A})$

$H(\mathcal{C}_n|\mathcal{A}) \nearrow H(\mathcal{C}|\mathcal{A})$

Proof: ① $[x]_{\mathcal{C}_2} \subset [x]_{\mathcal{C}_1}$ ✓

② $\mathcal{C} = \sigma(q)$ $\quad q$ finite

$$I_\mu(q|\mathcal{A})(x) = \sum_{P \in q} \mathbb{1}_P(x) \, (-\log \mu_x^{\mathcal{A}}(P))$$

$q_k \nearrow \mathcal{C}$ take limits which exists by monotone conv

③

$$I_\mu(q|\mathcal{A})(x) = -\log \mu_x^{\mathcal{A}}([x]_q)$$

$$= -\mathbb{1}_{[x]_q}(x) \log \mu_x^{\mathcal{A}}([x]_q)$$

$$+\,0 = \sum_{P \not\ni x} \mathbb{1}_P(x) \, -\log \mu_x^{\mathcal{A}}(P)$$

$$③ \quad [x]_{\mathcal{C}} = \bigcap_k [x]_{\mathcal{C}_k}$$

Lemma: $H(\mathcal{C}|\mathcal{A}) = \int H_{\mu_x^{\mathcal{A}}}(\mathcal{C})\, d\mu$

Proof: $\mathcal{C} = \sigma(\mathcal{Q})$ $\quad \mathcal{Q}$ finite

$$H(\mathcal{C}|\mathcal{A}) = \int I(\mathcal{Q}|\mathcal{A})(x) \, d\mu(x)$$

$$= \int \sum_{P \in \mathcal{Q}} \mathbb{1}_P(x) \, -\log \mu_x^{\mathcal{A}}(P) \quad d\mu(x)$$

$$= \int d\mu(y) \int d\mu_y^{\mathcal{A}}(x) \sum_{P \in \mathcal{Q}} \mathbb{1}_P(x) \, -\log \mu_{\cancel{x}\, y}^{\mathcal{A}}(P)$$

$$= \int d\mu(y) \sum_{P \in \mathcal{Q}} -\log \mu_y^{\mathcal{A}}(P) \int d\mu_{\cancel{x}}^{\mathcal{A}}(x) \, \mathbb{1}_P(x)$$

$$= \int d\mu(y) \sum_P -\log \mu_y^{\mathcal{A}}(P) \; \mu_y^{\mathcal{A}}(P)$$

$= \int d\mu(y) \; H_{\mu_y^{A}}(\xi)$

General case $\xi_k \nearrow \mathcal{C}$

and use monotonicity

(③ of Prop)

<u>Prop</u>: $A_n \nearrow A$, $A_n \searrow A$

① ξ countable $\quad I_\mu(\xi | A_n) \to I_\mu(\xi|A)$

② ξ finite $\quad H_\mu(\xi|A_n) \to H(\xi|A)$

<u>Proof</u>: Martingale convergence thm

$A_n \nearrow A \quad \mathbb{E}[f|A_n] \nearrow \mathbb{E}[f|A]$

$A_n \searrow A \quad \mathbb{E}[f|A_n] \searrow \mathbb{E}[f|A]$

① follows from this and $-\log$ cont

$x \in P \in \xi \quad I_\mu(\xi|A_n) = -\log \mathbb{E}[\mathbb{1}_P|A_n](x)$

② $H_{\mu_x^{A_n}}(q) \to H_{\mu_x^{A}}(q)$ $\mu_x^{A_n}(P)$

$\nearrow$

Σ $\phi(x) = -x \log x$

Lemma

$\lim H_\mu(q|A_n) = \int \lim H_{\mu_x^{A_n}}(q) \, d\mu$

$H_\nu(q) \le \cancel{\alpha}\, q$

Prop: (Monotonicity (Subadditivity))

① $I(e_1 \vee e_2 | A) = I(e_1|A) + I(e_2|A \vee e_1)$

$H(e_1 \vee e_2|A) = H(e_1|A) + H(e_2|A \vee e_1)$

② $H(e_1|e_2 \vee A) \le H(e_1|A)$

③ $H(e_1 \vee e_2|A) \le H(e_1|A) + H(e_2|A)$

$q, A_n \nearrow A$

Prop: ① $\int \sup_n I(\xi|A_n)(x)\,d\mu < \infty$

② ξ countable $H_\mu(\xi|A_n) \nearrow H_\mu(\xi|A)$

Proof: ① ⇒ ② down conv

① $I^*(x) = \sup_n I(\xi|A_n)(x)$

$A = \{(x,t) : I^*(x) > t\} \subset X \times \mathbb{R}_{\geq 0}$

Fubini

$$\iint_A d\mu\, dt = \int d\mu\, I^*(x)$$
$$= \int dt\, \mu\{x : I^*(x) > t\}$$

$$\mu(x : I^*(x) > t) = \sum_{P \in \xi} \mu\{x \in P : I^* > t\}$$
$$= \sum_{P \in \xi} \mu\{x \in P : \inf_n \mu_x^{A_n}(P) < e^{-t}\}$$
$$= \sum_{P \in \xi} \sum_n \mu\{x \in P : \mu_x^{A_m}(P) \begin{cases} < e^{-t} & n = m \\ \geq e^{-t} & m < n \end{cases}\}$$

$\overset{\text{def}}{=} A_n(P) \in A_n$

$$= \sum_P \sum_n \mu(P \cap A_n(P))$$

(3) cond. meas

$$= \sum_P \sum_n \int_{A_n(P)} d\mu(x) \int \mathbb{1}_P(y)\, d\mu_x^{A_n}(y)$$

$$= \sum_P \sum_n \underbrace{\int_{A_n(P)} d\mu(x)\, \mu_x^{A_n}(P)}_{\leq e^{-t}}$$

$$\leq \sum_P \min\Big(\underbrace{\sum_n e^{-t} \mu(A_n(P))}_{=:\mu(P)}, \mu(P) \Big)$$

$$\boxed{\leq \sum_P \min(e^{-t}, \mu(P))}$$

$$\sum_P \int dt \min(e^{-t}, \mu(P))$$

$$\sum_P \int_0^{-\log \mu(P)} \mu(P)\, dt + \int_{-\log \mu(P)}^{\infty} e^{-t}\, dt$$

$$\geq \sum_{P} -\mu(P)\log\mu(P) + \mu(P)$$

$$= H_\mu(Q) + 1 \qquad \blacksquare$$

Cor: Q 1-sided generator

$$Q_0^n \nearrow \mathcal{B}(X)$$

$$\cancel{H} \quad h_\mu(T) = h_\mu(T, Q)$$

$$= \lim_{n\to\infty} H_\mu(Q \mid Q_1^n)$$

$$= H_\mu(Q \mid Q_1^\infty) \qquad \blacksquare$$

Resume in two weeks

Dates : Resume 16.4