Lecture 14

$X = G/\Gamma = SL_2(\mathbb{R})/SL_2(\mathbb{Z})$

$u_s = \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix}$

$a_t = \begin{bmatrix} e^{t/2} & \\ & e^{-t/2} \end{bmatrix}$

δ_{Λ_q} normalized counting measure

$\Lambda_q \subset (\mathbb{Z}/q\mathbb{Z})^\times$

$$\delta_{\Lambda_q} = \frac{1}{\#\Lambda_q} \sum_{p \in \Lambda_q} \delta_{u_{\frac{p}{q}}\Gamma}$$

$$a_t u_{\frac{p}{q}} \mathbb{Z}^2 = \begin{bmatrix} e^{t/2} & e^{t/2}\frac{p}{q} \\ & e^{-t/2} \end{bmatrix} \mathbb{Z}^2$$

$\ni e_1$

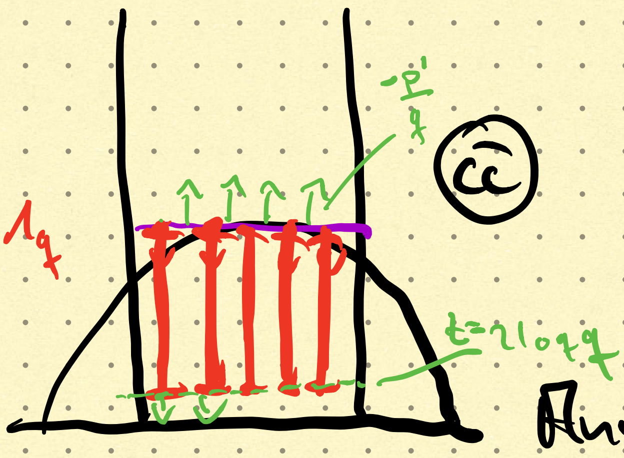
$\begin{pmatrix} e^{t/2} \\ 0 \end{pmatrix}$ $t \to -\infty$ diverges

$\ni \begin{pmatrix} p \\ -q \end{pmatrix}$

$\begin{pmatrix} 0 \\ -e^{-t/2} q \end{pmatrix}$ diverge $t \geq 2\log q$

Thm 1 (David–Shapira)

Suppose (i) $\displaystyle\lim_{q\to\infty} \frac{\log^* \Lambda_q}{\log q} = 1$



(ii) $\displaystyle \bar\mu_q := \frac{1}{2\log q}\int_0^{2\log q} a_t * \delta_{\Lambda_q}\, dt$

Any accumulation pt of $\bar\mu_q$ is a probability. ("No escape of mass")

then $\bar\mu_q \to$ Haar on X

Thm 2: (DS) (i) & (ii) holds for $\Lambda_q = \left(\frac{\mathbb{Z}}{q\mathbb{Z}}\right)^\times$

Instead looking $\bar\mu_q$ take

$$\mu_q = A_{\lfloor \log q \rfloor} * \delta_{\Lambda_q}$$

$$A_N f(x) = \frac{1}{N}\sum_{n<N} f(a^n . x)$$

$$N_q := \lfloor \log q \rfloor \qquad a = a_1$$

We prove Thm 1 w/ Γ_q

<u>Bowen ball</u>:

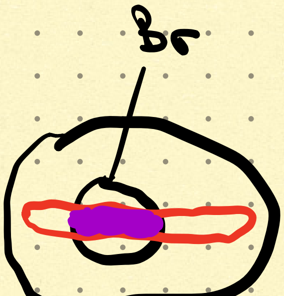
balls in X wrt metric

$$d_N(x,y) = \max_{n=0\ldots N-1} d(a^n . x, a^n . y)$$

If $r < \dfrac{\text{inj rad }(\text{at } x)}{\|Ad_a\|}$ then

N- Bowen ball of rad. r

$$\left(\bigcap_{n<N} a^{-n} B_r^G(e)\, a^n\right) . x$$



Make use $G = PU = \begin{bmatrix} * & \\ * & * \end{bmatrix} \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix}$

$P_\varepsilon = \{ p : \|p - e\|_\infty < \varepsilon \}$ ↑ (locally) unstable wrt a

$U_\varepsilon = \{ u_s : |s| < \varepsilon \}$

$$B_\varepsilon := P_\varepsilon U_\varepsilon \qquad \left(\supset B^G_{\frac{\varepsilon}{100}}(e) \right)$$

Associated Bowen-ball is

$$B_{\varepsilon, N} \cdot z = P_\varepsilon U_{\varepsilon e^{-(N-1)}} \cdot z$$

$$= \left(\bigcap_{n < N} a^{-n} B_\varepsilon a^n \right) \cdot z$$

<u>Separation</u>: $N_q = \lfloor \log q \rfloor$

<u>Lemma 1</u>: $\forall x$

$$\# \; B_{\frac{1}{100}, N_q} \cdot x \cap \{ u_{\frac{p}{q}} \Gamma : p \in \mathbb{Z}/q \}$$

$$\leq 1$$

Proof: Sps $u_{\frac{p_1}{q}}\Gamma, u_{\frac{p_2}{q}}\Gamma \in B_{\frac{1}{100}, N_q} \cdot x$

$x = g\Gamma$

$$\exists b_1, b_2 \in B_{\frac{1}{100}, N_q}$$

$$\gamma_1, \gamma_2 \in \Gamma$$

$$u_{\frac{p_1}{q}}\gamma_1 = b_1 g \qquad u_{\frac{p_2}{q}}\gamma_2 = b_2 g$$

$$u_{\frac{p_2}{q}}\gamma_2 = b_2 b_1^{-1} u_{\frac{p_1}{q}}\gamma_1$$

$\exists \gamma \in \Gamma$ $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ st $u_{\frac{p_2}{q}}\gamma\, u_{-\frac{p_1}{q}} = b$

$\exists b \in B_{\frac{1}{100}, N_q} \cdot B^{-1}_{\frac{1}{100}, N_q}$

$$\subset B_{\frac{1}{10}, N_q} \ni \begin{bmatrix} a + \frac{p_2}{q}c & b + \frac{p_2}{q}d - \frac{p_1}{q}\left(a + \frac{p_2}{q}c\right) \\ c = 0 & d - \frac{p_1}{q}c \end{bmatrix}$$

Lemma 4

$a + \frac{p_2}{q}c = 1$, $d - \frac{p_1}{q}c = 1$

$$\in P_{\frac{1}{10}} U_{\frac{1}{10}e^{-(N_q-1)}}$$

$$b + \frac{1}{q}(p_2 - p_1) \leq \frac{1}{10} e^{-(N_q-1)}$$

$\Rightarrow p_1 = p_2$ $\blacksquare$

Lemma 2: Sps $S: Y \to Y$ μ prob

ξ finite partition

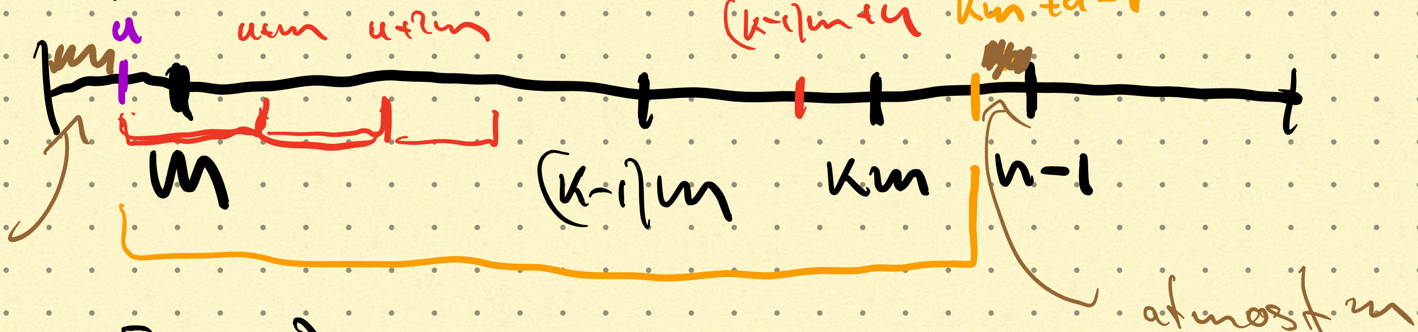
$A_{n*}\mu = \frac{1}{n} \sum_{k<n} S^i_* \mu$

then $m < n$

$$\frac{1}{m} H_{A_{n*}\mu}(\xi_0^{m-1})$$

$$\geq \frac{1}{n} H_\mu(\xi_0^{n-1}) - \frac{m}{n} \log \# \xi$$

Proof: $n-1 = km + r$



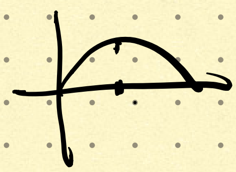
$u \in [0, m)$

$$H_\mu(\xi_0^{n-1}) \le \underbrace{\sum_{i=0}^{k-1} H_\mu\left(S^{-(im+u)}\xi_0^{m-1}\right)}$$

$$+ \underbrace{m \log \#\xi}$$

Sum over $u \in [0,m)$

$$m H_\mu(\xi_0^{m-1}) \le \sum_{i=0}^{km-1} H_\mu\left(S^{-i}\xi_0^{m-1}\right) + m^2 \log \#\xi$$



$\alpha(x) = -x\log x$

$\frac{1}{n}\sum_i \sum_P \alpha(S^i_*\mu(P))$ Jensen

$\le \sum_P \alpha\left(\frac{1}{n}\sum_i S^i_*\mu(P)\right)$

$$\le \frac{n}{n}\sum_{i=0}^{n-1} H_{S^{i}_*\mu}\left(\xi_0^{m-1}\right) + m^2 \log \#\xi$$

$$\le n\, H_{A_n * \mu}\left(\xi_0^{m-1}\right) + m^2 \log \#\xi$$

■

Cor: Suppose $\mu_q = A_{N_q} * \delta_{\Lambda_q} \xrightarrow{w^*} \mu$ accum.

Sps $\mu(\partial \xi) = 0$

then $$h_\mu(a, \xi) \geq \lim_{q = q_k} \frac{1}{N_q} H_{\delta_{\Lambda_q}}(\xi_0^{N_q - 1})$$

Proof:

Recall

$\nu_n \xrightarrow{w^*} \nu$ $\qquad$ $\nu_n(B) \to \nu(B)$

$\qquad$ if $\nu(\partial B) = 0$

$H_{\nu_n}(\xi) \to H_\nu(\xi)$

$$h_\mu(a, \xi) \overset{\text{Def.}}{=} \lim_m \frac{1}{m} H_\mu(\xi_0^{m-1})$$

$$\overset{\text{Recall}}{=} \lim_m \lim_q \frac{1}{m} H_{\mu_q}(\xi_0^{m-1})$$

$$\overset{\text{Lem 2}}{\geq} \lim_m \lim_q \left(\frac{1}{N_q} H_{\delta_{\Lambda_q}}(\xi_0^{N_q - 1}) - \frac{m}{N_q} \log \# \xi \right)$$

$$= \lim_q \frac{1}{N_q} H_{\delta_{\Lambda_q}}(\xi_0^{N_q - 1})$$

$\blacksquare$

Suppose $\exists r \quad \forall P \in \xi_0^{N_q - 1}$

$$\left| P \cap \{ a_{\substack{p\\q}} \Gamma : p \in \Lambda_q \} \right| \leq r$$

i.e. $$\delta_{\Lambda_q}(P) \leq \frac{r}{\#\Lambda_q}$$

then $$\frac{1}{N_q} H_{\delta_{\Lambda_q}}\left(\xi_0^{N_q-1}\right) = \frac{1}{N_q} \sum_P \delta_{\Lambda_q}(P) \log \frac{1}{\delta_{\Lambda_q}(P)}$$

$$\geq \frac{1}{N_q} \sum_P \delta_{\Lambda_q}(P) \left(\log \#\Lambda_q - \log r \right)$$

$$= \frac{\log \#\Lambda_q}{N_q} - \frac{\log r}{N_q}$$

$$\downarrow \qquad\qquad \downarrow$$

by ☺ $\quad 1 \qquad\qquad 0$

Hence ≥ 1 $q \to \infty$

By the cor for any accumpt μ (by (cc) is prob)

$$h_\mu(a, \mathcal{Q}) \geq 1 = h_{Haar}(a)$$

$$\Rightarrow \mu = Haar$$

Take $\mathcal{Q}$ st $\forall P \in \mathcal{Q}$ finite $P \subset B_\eta \cdot X$

then $P \in \mathcal{Q}_0^{n-1}$ $P \subset B_{\eta, n} \cdot X$

But: X non cmpct, no such finite $\mathcal{Q}$ exists?

$X^{\leq M} = \{ x \in X : \text{shortest vector has norm} \geq \frac{1}{M} \}$

$X^{>M} = (X^{\leq M})^c$ cusp

Def: $\mathcal{E} = \{ P_0, P_1, \dots, P_n \}$

$P_0 = X^{>M}$

$i>0$ $\quad P_i \subset B_n \cdot x_i \qquad x_i \in X^{\leq M}$

call $\mathcal{E}$ a (M, n)-partition

Rem: Fixing some μ, there exists w/ $\mu(\partial \xi) = 0$

Lemma 3: (ELMV on Duke Thm) $\forall M \quad \forall \eta \ll_M 1$

$\forall \left(M, \frac{\eta}{10}\right)$ partitions $\mathcal{P}$ we have

$\forall \kappa \in (0,1), \forall N \geq 1$

$\exists X' \subset X^{\leq M}$ such that

① $X' = \bigsqcup_{i \in I} S_i \qquad S_i \subset \mathcal{P}_0^{N-1}$

② $\forall i \in I \quad S_i \subset \bigcup_{j \in J} B_{\eta, N} z_j$

where $z_j \in S_i$ and $|J| \leq C^{\kappa N}$

where C is an absolute constant

③ $\forall$ prob μ

$$\mu(X') \geq \mu(X^{\leq M}) - \frac{1}{\kappa} A_N * \mu(X^{>M})$$

Proof of ① & ③:

$$f(x) = A_N\left(\mathbb{1}_{X^{>M}}\right)(x)$$

f $\mathcal{P}_0^{N-1}$-measurable

$X' = X^{\le M} \cap f^{-1}([0,K])$

$= \{$ pts $x^{\le M}$ st κN of their $[0,N]$ orbit is inside comp $\}$

ξ_0^{N-1}-measurable $\Rightarrow$ (1)

$$\mu(X') \ge \mu(X^{\le M}) - \mu(f > K)$$

$$\ge \mu(X^{\le M}) - \frac{1}{K}\mu(f)$$

$$= \mu(X^{\le M}) - \frac{1}{K} A * \mu(X^{>M})$$

$\Rightarrow$ (3)

∎

<u>Proof Thm 1</u>: μ any acc pt of μ_q

Need to show $\forall \varepsilon \; \exists \xi \;\; h_\mu(a,\xi) \ge 1-\varepsilon$

Let ξ $(M, \frac{n}{10})$ partition, K fixed apply Lemma 3 to $\mathcal{N} = N_q$

get some X'

2 $\forall P \in \mathcal{P}_0^{N_q - 1}$ $P \subset X'$

P covered by $\leq C^{\varkappa N_q}$ BBalls

Separation Lemma $\longrightarrow \{ B_{\varkappa, N_q}.x_i \}$

each such ball has $\delta_{\Lambda_q}(B) \leq \frac{1}{\# \Lambda_q}$

$$\Rightarrow \delta_{\Lambda_q}(P) \leq \frac{C^{\varkappa N_q}}{\# \Lambda_q}$$

$$h_\mu(a, \mathcal{P}) \overset{\text{Cor}}{\geq} \lim_q \frac{1}{N_q} H_{\delta_{\Lambda_q}}\left(\mathcal{P}_0^{N_q - 1}\right)$$

$$= \lim_q \frac{1}{N_q} \sum_{P} \delta_{\Lambda_q}(P) \log \frac{1}{\delta_{\Lambda_q}(P)}$$

$\geq$ ($P \subset X'$)

$$\geq \lim_q \delta_{\Lambda_q}(X') \frac{\log \# \Lambda_q - \varkappa N_q \log C}{N_q}$$

$$\geq \lim_q \left(\underbrace{\delta_{1_q}(X^{\leq M})}_{=1} - \frac{1}{\kappa} \overset{\mu}{\overset{\uparrow}{\mu_q}}(X^{>M}) \right) \left(\underbrace{\frac{\log \# 1_q}{N_q}}_{\to 1} - \kappa \log C \right)$$

$$= \left(1 - \frac{1}{\kappa} \mu(X^{>M})\right)\left(1 - \kappa \log c\right)$$

κ small

M large

$$\geq 1 - \varepsilon \quad \Rightarrow \quad \mu = \text{Haar} \qquad \blacksquare$$

Still need to prove (2) of Lemma 3.

Lemma 4: $H < G$ $\quad H_\varepsilon = \{h \in H : \|h - e\|_\infty < \varepsilon\}$

① $H_\varepsilon = H_\varepsilon^{-1}$ $\quad \ulcorner \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \lrcorner$

② $H_{\varepsilon_1}, H_{\varepsilon_2}$ $\quad \varepsilon_i < 1$ then

$$H_{\varepsilon_1} \cdot H_{\varepsilon_2} \subset H_{2(\varepsilon_1 + \varepsilon_2)}$$

$$\ulcorner (e + X_1)(e + X_2) = e + \underset{\le \varepsilon_1}{X_1} + \underset{\le \varepsilon_2}{X_2} + \underbrace{X_1 X_2}_{\substack{\le 2\varepsilon_1\varepsilon_2 \\ \le \varepsilon_1^2 + \varepsilon_2^2 \\ \le \varepsilon_1 + \varepsilon_2}} \lrcorner$$

③ $U_{\varepsilon^+} P_{\varepsilon^-} \subset P_{2\varepsilon^-} U_{2\varepsilon^+}$ $\quad \varepsilon^\pm \le \frac{1}{4}$

$\ulcorner\ up = \tilde{p}\tilde{u} \lrcorner$

④ $\forall g \in P_{\varepsilon^-}$ $\quad g U_{\varepsilon^+} g^{-1} \subset P_{6\varepsilon^-} U_{2\varepsilon^+}$

$$\ulcorner \text{LHS} \subset P_{\varepsilon^-} U_{\varepsilon^+} P_{\varepsilon^-}^{-1} \subset P_{\varepsilon^-} P_{2\varepsilon^-} U_{2\varepsilon^+}$$

$$\subset P_{6\varepsilon^-} U_{2\varepsilon^+} \lrcorner$$

⑤ $x, y \in X \quad y \in P_{\varepsilon^-} U_{\varepsilon^+} \cdot x$

then $P_{\varepsilon^-} U_{\varepsilon^+} \cdot x \subset P_{6\varepsilon^-} U_{8\varepsilon^+} \cdot y$

$\ulcorner P_{\varepsilon^-} U_{\varepsilon^+} \cdot x \subset P_{\varepsilon^-} U_{\varepsilon^+} U_{\varepsilon^+} P_{\varepsilon^-} \cdot y$

$4\varepsilon^+$

$2\varepsilon^- \qquad 8\varepsilon^+$

$6\varepsilon^- \qquad \lrcorner$

Proof Lemma 5 part ② :

$P \in \mathcal{P}_0^{N-1} \qquad P \subset X'$

$V_m = \#\{ i < m \quad a^i P \subset X^{>m} \}$

Claim: $P \subset \bigcup_{j \in J} B_{\eta, m} \cdot y_j \qquad y_i \in P$

$\forall m \leq N \qquad |J| \leq C^{|V_m|}$

② follows from $m = N-1$.

Prove Claim by induction:

$m=1$: $y \in \underset{\mathcal{P}_0^{N-1}}{\underset{\cap}{P}} \subset \underset{\mathcal{Q}}{\underset{\cap}{P_i}} \subset B_{\frac{\eta}{10}} \cdot x_i \subset B_\eta \cdot y$

$q(m, \frac{m}{10})$

$m \to m+1$:

$m < N$

Hypothesis $y_j \in P$

$$P \subset \bigcup_{j \in J} B_{\eta, m} \cdot y_j$$

$$|J| \leq C^{|V_m|}$$

2 cases:

- $a^m P \subset X^{\leq m}$ (so $V_m = V_{m+1}$)

subclaim $\forall j$: $P \cap B_{\eta, m} \cdot y_j$

$$= P \cap B_{\eta, m+1} \cdot y_j$$

$\Rightarrow$ Finish this case

Proof subclaim: $y_j \in P$

$$a^m y_j \subset a^m \underset{\substack{\cap \\ \mathcal{G}_0^{m-1}}}{P} \subset \underset{\substack{\uparrow \\ \mathcal{G}}}{P_i} \subset B_{\frac{\eta}{10}} \cdot x_i$$

$$\subset B_{\eta_0} \cdot a^m y_j$$

Sps $g y_j \in P \cap B_{\eta, m} \cdot y_j$

$$a^m g y_j = a^m g a^{-m} \underbrace{a^m y_j}_{\uparrow\, X^{<m}} \in a^m P \cap b_\eta a^m y_j$$

$$\cap\, a^m B_{\eta, m} a^{-m} a^m y_j$$

$\Rightarrow \quad a^m g a^{-m} \in B_\eta \cap a^m B_{\eta, m} a^{-m}$

$\Rightarrow \quad g \in B_{\eta, m+1}$

- 2^{nd} case: $a^m P \subset X^{>m}$

Lemma 5:

$\exists C \quad \forall \varepsilon^+, \varepsilon^- \ll 1 \quad \forall x \in X$

$\forall \cancel{Y} \subset P_{\varepsilon^-} U_{\varepsilon^+} \cdot x$

$$Y \subset \bigcup_{i \in I} P_{\varepsilon^-/e} U_{\varepsilon^+/e} \cdot y_i$$

$|I| \leq C$ $\blacksquare$

Assuming this, $P_\eta U_{\eta e^{-(m-1)}} \subset P_\eta U_{\eta e^{-m}}$
$\quad \quad \quad \quad \; \| {}^{/e} \quad\quad {}^{/e}$

cover $P \cap B_{\eta, m} \cdot y_j$

$\subset \bigcup_k B_{\eta, m+1} \cdot y_k$

$C \sim$ many

Apply to all x_j

cover P $\quad C^{|V_m|} \cdot C$

$= C^{|V_{m+1}|}$ $\blacksquare$