Lecture 13

Recall: Parry measure maximizes entropy of a Markov shift.

Eg simple directed strongly connected graph on $V = \{1, \dots, m\}$

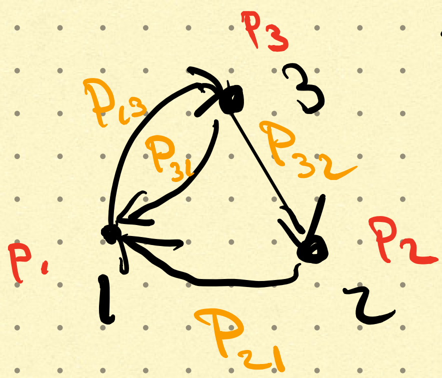
A Adjacency matrix

$$A_{ij} = 1 \quad \text{iff} \quad i \overset{\exists\,\text{edge}}{\rightsquigarrow} j$$

Dynamical system: $X = V^{\mathbb{N}}$, $T =$ shift

"space of paths"

$$\underline{x} = (x_0, x_1, x_2 \dots)$$



$\underline{p} = (p_1, \dots, p_m)$ prob vect

$P = (P_{ij})$ stochastic matrix

Markov measure

$$\mu_{\underline{p},P}\left([a_n, \dots, a_{n+s}]_n^{n+s}\right) = p_{a_n} \prod_{j=n}^{n+s-1} P_{a_j a_{j+1}}$$

Parry measure: Perron Frobenius says

! maximal Eigval λ

and left/right Eig vec u, v

st $\underline{u} A = A \underline{v} = \lambda A$

$$p_i = u_i v_i$$

$$P_{ij} = \frac{A_{ij} v_j}{\lambda v_i}$$

$\Rightarrow$ $m = \mu_{p,P}$ Parry meas

$$h_m(T) = \log \lambda$$

If ν prob, T-inv st $h_\nu(T) \geq h_m(T)$

$\Rightarrow \nu = m$

Today: Einsiedler Inequality

Thm: (Kadyrov)

If $f: X \to \mathbb{R}$ Lipschitz, $\forall \nu$ T-erg prob

$$|m(f) - \nu(f)| \ll \mathrm{Lip}(f) \left(h_m(T) - h_\nu(T)\right)^{1/2}$$

- Metric on X: $d(\underline{x}, \underline{y}) = \min\limits_{n : x_n \neq y_n} \frac{1}{2^n}$

f is Lipschitz w/ $\mathrm{Lip}(f) = L$

$$|f(x_0, \dots, x_n, \underline{z}) - f(x_0, \dots, x_n, \underline{z}')| \leq L\, 2^{-(n+1)}$$

Assume

- $m(f) = 0 \Rightarrow \|f\|_\infty \leq L$

<u>Transfer operator</u>

$f, g \in L^2_m(X)$ $\quad \langle f \circ T, g \rangle = \langle f, T^* g \rangle$

T^*: Adjoint of $\circ T$

<u>Properties</u>:

- $T^* g = \dfrac{d\, T_* m_g}{d m}$ $\quad dm_g = g\, dm$ $\quad (g \geq 0)$

Since $\langle f \circ T, g \rangle = \int f \circ T \; g \, dm$

$$= m_g(f \circ T)$$

$$= T_* m_g(f)$$

$$= \int f \, \frac{dT_* m g}{dm} \, dm$$

$$= \langle f, \frac{dT_* m g}{dm} \rangle$$

- $T^*(g \circ T) = g$
- $(T^* g) \circ T = \mathbb{E}_m [g \mid T^{-1} \mathcal{B}_X]$

Γ sps f $T^{-1}\mathcal{B}_X$-meas.
$(\Leftrightarrow \exists \tilde{f} \quad f = \tilde{f} \circ T)$

$$\langle (T^* g) \circ T, f \rangle = \langle (T^* g) \circ T, \tilde{f} \circ T \rangle$$

$$= \langle g, \tilde{f} \circ T \rangle$$

$$= \langle g, f \rangle$$

$\rightsquigarrow$ defing property

prop of cond expect. $\dashv$

• Thm (Ruelle)

T^* has a spectral gap

$T^{*n} f \to m(f)$ exponentially fast

Proof for Bernoulli shift:

(X, T, μ) $\quad \mu = \left(\sum \delta_i p_i\right)^{\otimes \mathbb{N}}$

$$T^* f(\underline{x}) = \frac{d\, T_* \mu_f}{d\mu}(\underline{x})$$

$$= \lim_n \frac{T_* \mu_f([x_0, \dots, x_n]_0^n)}{\mu([x_0, \dots, x_n]_0^n)}$$

$$= \lim \frac{\mu_f(T^{-1}[x_0, \dots, x_n])}{\mu([x_0, \dots, x_n])}$$

$$= \lim_n \frac{\mu_f\left(\bigcup_z [z, x_0, \dots, x_n]\right)}{\mu([x_0, \dots, x_n])}$$

f $\mathcal{G}_0^m$-meas $\quad = \lim_n \dfrac{\sum_z f(z,\underline{x})\,\mu([z,x_0,\dots,x_n])}{\mu([x_0,\dots,x_n])}$

$\mathcal{G} = \{[\varepsilon]_0\}$

$m \to \infty$

$$= \sum_z f(z,\underline{x})\, p_z$$

$$\mathrm{Lip}(T^* f) = \frac{\mathrm{Lip}(f)}{2}$$

(b/c

$T^* f(x_0,\dots,x_n,\underline{y}) - T^* f(x_0,\dots,x_n,\underline{y}')$

$$= \sum_z \Big(f(z,x_0,\dots,x_n,\underline{y}) - f(z,x_0,\dots,x_n,\underline{y}')\Big)\, p_z$$

$$\le \sum_z \mathrm{Lip}(f)\, 2^{-(n+2)}\, p_z$$

$$= \frac{\mathrm{Lip}(f)}{2}\, \frac{1}{2^{(n+1)}}$$

)

$\mu(f)=0$

$$\mathrm{Lip}(T^{*n} f) \leq \frac{\mathrm{Lip} f}{2^n}$$

$\leq$

$\|T^{*n} f\|_\infty$

- $I_m(\mathcal{Q} \mid \mathcal{Q}_1^\infty)(\underline{x})$

$$= -\log m_{\underline{x}}^{\mathcal{Q}_1^\infty}([\underline{x}]_{\mathcal{Q}})$$

$$h_m(T) = H_m(\mathcal{Q} \mid \mathcal{Q}_1^\infty)$$

$$= \int dm \; I_m(\mathcal{Q} \mid \mathcal{Q}_1^\infty)$$

Also $\underline{\forall}$ ν T-inv prob

$$h_m(T) = \int d\nu \; I_m(\mathcal{Q} \mid \mathcal{Q}_1^\infty)$$

<u>Proof</u>: $m([a_0, \ldots, a_n])$

$$= u_{a_0} V_{a_n} \lambda^{-n}$$

$$m_{\underline{x}}^{q_1^{\infty}}([x]_q) = \lim_n \frac{m([x]_q \cap [x]_{q_1^n})}{m([x]_{q_1^n})}$$

$q_1^n \nearrow q_1^{\infty}$

$$= \lim_n \frac{m([x_0, \ldots, x_n]_0^n)}{m([x_1, \ldots, x_n]_1^n)}$$

$$= \frac{u_{x_0} V_{x_n} \lambda^{-n}}{u_{x_1} V_{x_n} \lambda^{-(n-1)}}$$

$$= \frac{u_{x_0}}{u_{x_1}} \lambda^{-1}$$

$I_m(q|q_1^{\infty})(x)$

$$= -\log \frac{u_{x_0}}{u_{x_1}} \lambda^{-1} = \log \lambda + (\log u \circ T - \log u)$$

where $u(\underline{x}) = u_{x_0}$

Integrate wrt $\mathcal{T} - \tilde{\sim} v$ $\mathcal{D}$

$$= \log 1$$

$\perp$

Recall:

Pinsker inequality

p, q prob vectors

$D(p \| q)$ "relative entropy"

$\|$ "KL-divergence"

$$\sum_i p_i \log \frac{p_i}{q_i}$$

$$\|p - q\|_{\ell^1}^2 \leq 2\, D(p \| q)$$

Proof of Ruinsiedler Inequality

$m(f) = 0 \qquad f_n = (T^*)^n f \to 0$

exp fast

$$\|f_n\|_\infty \leq \frac{\mathrm{Lip}(f)}{2^n}$$

$$|m(f) - \nu(f)| = |\nu(m(f) - f)|$$

$$= \lim_{n\to\infty} |\nu(f_n - f_0)|$$

$$\leq \lim_n \left| \sum_{k=0}^{n-1} \nu(f_{k+1} - f_k) \right|$$

$$\leq \sum_{n=0}^{\infty} |\nu(f_{n+1} - f_n)|$$

$$f_{n+1} = T^* f_n \qquad (T^* f_n) \circ T$$

$$= \mathbb{E}_m[f_n | \mathcal{F}_1^\infty]$$

$$\nu\left(f_{n+1}^{OT} - f_n\right) = \nu\left(\mathbb{E}_m\left[f_n \mid \mathcal{G}_1^{\infty}\right] - \mathbb{E}_\nu\left[f_n \mid \mathcal{G}_1^{\infty}\right]\right)$$

$$= \nu\left(m_x^{\mathcal{G}_1^{\infty}}(f_n) - \nu_x^{\mathcal{G}_1^{\infty}}(f_n)\right)$$

$$m_{\underline{x}}^{\mathcal{G}_1^{\infty}} = \sum_z m_{\underline{x}}^{\mathcal{G}_1^{\infty}}\Big|_{[z]_0}$$

$$= \nu\left(\sum_z f_n(z, x_1, x_2 \ldots) \cdot \left(\underbrace{m_x^{\mathcal{G}_1^{\infty}}([z]_0)}_{\overset{\text{def}}{=} q_x(z)} - \underbrace{\nu_x^{\mathcal{G}_1^{\infty}}([z]_0)}_{= p_x(z)}\right)\right)$$

$$\sum_{V \ni z} q_x(z) = 1 = \sum_{z \in V} p_x(z)$$

$$\leq \nu\left(\|f_n\|_{\infty} \sum_z \left| q_x(z) - p_x(z) \right|\right)$$

$$\overset{CS}{\leq} \|f_u\|_\infty \sqrt{\nu\left(\|q_x - p_x\|^2_{\ell^1(W)}\right)}$$

$$\overset{\text{Pinsker}}{\leq} \|f_u\|_\infty \sqrt{\nu\left(2 D(p_x \| q_x)\right)}$$

$$\nu\left(D(p_x \| q_x)\right)$$

$$= \int d\nu \sum_{P \in \xi} \nu_x^{q_1^\infty}(P) \log \frac{\nu_x^{q_1^\infty}(P)}{m_x^{q_1^\infty}(P)}$$

$$= \int d\nu \sum_P \nu_x^{q_1^\infty}(P) \log \nu_x^{q_1^\infty}(P)$$

$$- \int d\nu \sum_P \nu_x^{q_1^\infty}(P) \log m_x^{q_1^\infty}(P)$$

$$= \int d\nu(x) \int d\nu_x^{q_1^\infty}(y) \log \nu_y^{q_1^\infty}([y]_\xi)$$

$$- \int d\nu \int d\nu_x^{q_1^\infty}(y) \log m_y^{q_1^\infty}([y]_\xi)$$

$$= \nu\left(\log \nu_x^{q_1^\infty}([x]_q) - \log m_x^{q_1^\infty}([x]_q)\right)$$

$$= -h_\nu(T) + h_m(T)$$

$$\Rightarrow \quad \nu(f_{n+1} - f_n) \leq \|f_n\| \sqrt{2} \sqrt{h_m(T) - h_\nu(T)}$$

$$\Rightarrow \quad |m(f) - \nu(f)| \leq \left(\sum_n \|f_n\|\right) \sqrt{2} \left(h_m(T) - h_\nu(T)\right)^{\frac{1}{2}}$$

$$\sum_n \|f_n\| \leq \mathrm{Lip}(f) \sum \frac{1}{2^{n+1}} = 2\,\mathrm{Lip}(f)$$

$\blacksquare$

__Homogeneous setting:__

$a \in G \quad a \curvearrowright G/\Gamma \quad m$ Haar

we proved $\quad G_a^- = \{ g : a^n g a^{-n} \to e \}$

$$G = \langle G_a^-, G_a^+ \rangle$$

__then__ $h_m(a) = -\log \operatorname{mod}_{\mathfrak{g}_a^-}(a)$

$A \quad G_a^-$-subordinable & a^{-1}-decreasing

$[x]_A = V_x \cdot x \qquad V_x$ open bnd $\ni e$, $\subset G_a^-$

$a^{-n}A$

$$[x]_{a^{-n}A} = a^{-n} * [a^n \cdot x]_A$$

$$= a^{-n} V_{a^n \cdot x} \cdot (a^n \cdot x)$$

$$= \left(a^{-n} V_{a^n \cdot x} a^n\right) \cdot x$$

becomes long piece of G_a^--orbit

$f \in C_c^\infty(X)$

$$\left\{ \begin{array}{l} \mathbb{E}_m[f \mid a^{-n} \mathcal{A}](x) \\ \text{tend equidistribute} \\ \text{exp. fast} \end{array} \right.$$

$f_n = \mathbb{E}_m[f \mid a^{-n} \mathcal{A}](a^{-n} . x)$

<u>Recursion formula:</u>

$f_{n+1} = \mathbb{E}_m[f_n \mid a^{-1} \mathcal{A}](a^{-1} x)$

$f_n \to m(f)$

$f_0 \approx f$

<u>Caveat:</u> Speed of equidistribution depend on $[x]_{\mathcal{A}} = V_x . x$