Lecture 11

Thm B: $X = G/\Gamma$ $\quad a \in G$ $\quad \mu$ a-inv erg prob on X

$U < G_a^- = \{ g : a^n g a^{-n} \to e \}$

& $aUa^{-1} < U$

Then

$$h_\mu(a, U) \leq -\log \mathrm{mod}_a(U)$$

$$= h_{Haar}(a, U)$$

"=" iff μ is U-inv.

$$h_\mu(a, U) = \int d\mu \lim_{n\to\infty} \frac{1}{n} \log \mu_x^U (a^{-n} B_1^U a^{n})$$

$$\mathrm{mod}_U(a) = \frac{m_U(a B_1^U a^{-1})}{m_U(B_1^U)}$$

Cor: $a_t = \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} \curvearrowright SL_2(\mathbb{R})/\Gamma$

$$h_\mu(a) = 2t \iff \mu \text{ is Haar}$$

Pf By Thm B, μ is G_a^--inv

$G_{a^{-1}}^- = G_a^+$-inv $\quad \perp$

Proof of Thm B:

$\mathcal{A}$ $\mathcal{U}$ subordinate ; a - decreasing

$\Rightarrow h_\mu(a,\mathcal{U}) = H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})$

Last time

μ a.e. x $\exists \delta_x > 0$, $V_x \subset \mathcal{U}$ st

$[x]_{\mathcal{A}} = V_x \cdot x$, $B^{\mathcal{U}}_\delta \subset V_x \subset B^{\mathcal{U}}_{1/\delta}$

Fix $\delta > 0$ let $Y = \{ x$ st $\uparrow$ holds $\}$

$[x]_{a^{-1}\mathcal{A}} = (a^{-1} V_{a.x}\, a) \cdot x \supset [x]_{\mathcal{A}}$

$= \bigsqcup_{i \in \mathbb{N}} [x_i]_{\mathcal{A}} \sqcup N_x$ $\leftarrow$ $\mu_x^{a^{-1}\mathcal{A}}$ null set

Key prop:

$$\mu_x^{a^{-1}\mathcal{A}} = \frac{1}{\mu_x^{\mathcal{U}}(a^{-1}V_{a.x}\,a)} \mu_x^{\mathcal{U}}\Big|_{a^{-1}V_{a.x}\,a} \cdot x$$

compare

$$\mathcal{I}_x^{\mathrm{Haar}} = \frac{1}{m_{\mathcal{U}}(a^{-1}V_{a.x}\,a)} \, m_{\mathcal{U}}\Big|_{a^{-1}V_{a.x}\,a} \cdot x$$ prob. on $[x]_{a^{-1}\mathcal{A}}$

Observe:

$$\int d\underline{\underline{\mu}}(x) \log \Gamma_x^{\text{Haar}}([x]_A) = \log \operatorname{mod}_u(a)$$

$$\Gamma_{\Gamma_x}^{\text{Haar}}([x]_A) = \frac{m_u(V_x)}{m_u(a^{-1}V_{a.x}a)}$$

$$= \frac{m_u(V_x)}{m_u(V_{a.x})} \operatorname{mod}_u(a)$$

$$= f(x)$$

$$\frac{1}{n}\sum_{k<n} \log f(a^k.x) \xrightarrow[\text{Thm}]{\text{Erg}} \int \log f$$

$$\parallel$$

$$\log \operatorname{mod}_u(a) + \frac{1}{n}\Big(\underbrace{\log m_u(V_x) - \log m_u(V_{a^n.x})}_{\text{bnd along}}\Big)$$

$a \in X$, $a^n.x \in Y$ $\rightsquigarrow$ ∞-many n

$$\downarrow$$

$$\log \operatorname{mod}_u(a)$$

$\square$

$$-\log \mathrm{mod}_U(a) - h_\mu(a, U)$$

$$= -\int \log \tau_x^{Haar}([x]_A)\, d\mu - \int -\log \mu_x^{a^{-1}A}([x]_A)\, d\mu$$

$$= -\int \log \frac{\tau_x^{Haar}([x]_A)}{\mu_x^{a^{-1}A}([x])_A}\, d\mu$$

$$= -\int d\mu(x) \int_{[x]_{a^{-1}A}} d\mu_x^{a^{-1}A}(y) \log \frac{\tau_{\color{red}x}^{Haar}([y]_A)}{\mu_{\color{red}x}^{a^{-1}A}([y]_A)}$$

$$= -\int d\mu \sum_{[x_i]_A} \log \frac{\tau_x^{Haar}([x_i]_A)}{\mu_x^{a^{-1}A}([x_i]_A)} \mu_x^{a^{-1}A}([x_i]_A)$$

Jensen

$$\geq - \int d\mu \log \underbrace{\sum_{x_i} \tau_x^{Haar}([x_i])}_{\leq 1}$$

$\color{red}{=}$

b/c τ_x^{Haar} prob on $[x]_{a^{-1}A} = \bigcup [x_i]_A \cup N_x$

≥ 0

$\color{red}{=}$

$\rightsquigarrow U \leq U$ of Thm B

Suppose "$=0$"

$\Rightarrow \forall i \quad \tau_x^{Haar}(|[x_i]_A|)$

$= \mu_x^{a^{-1}A}([x_i]_A)$

$\Downarrow$

$k, l \geq 0$

$H_\mu(a^k A \mid a^{-l} A) = (k+l) H_\mu(A \mid a^{-1} A)$

$= (k+l)^{-\log} \mathrm{mod}_U(a)$

U-orbit

$\rightsquigarrow$ $\mu_x^{a^{-l}A}$ gives same weight

depends the above

contracting

to $a^k A$ atoms as

$\{e\} \cdot x$ 2

m_U would.

a.e. x, $a^l_{\cdot x}, a^{-k}_{\cdot}\, x \in Y$ ∞-often

$\Rightarrow \mu_x^U$ is Haar on U

$\Rightarrow \mu$ U-inv ∎

$T: X \to X$ cont, X cmpct

$$h_{top} = \sup_{\substack{\mathcal{U} \text{ open} \\ \text{cover of} \\ X}} \lim \frac{1}{n} \log \# \text{ of smallest subcover of } \mathcal{U}_0^{n-1}$$

Thm
$$= \sup_{\substack{\mu \text{ prob} \\ T\text{-inv}}} h_\mu(T) \qquad (*)$$

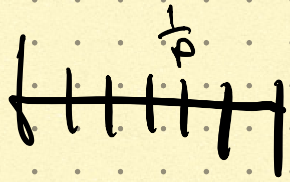
(X,T) intrinsically ergodic if $(*)$ is attained and is unique

E.g.: $a = \begin{bmatrix} a^t & \\ & a^{-t} \end{bmatrix}$, hyperbolic toral automorphism!

times p:

Times p map: $x \in \mathbb{T} \mapsto px \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ $\quad p$ integer

$\xi = \{ [\frac{i}{p}, \frac{i+1}{p}) \}_i$

ξ_0^{n-1}-atoms $\frac{1}{p^n}$-intervals 

* p^n - many of them

$$h_\mu(\times p) = h_\mu(\times p, \xi) = \inf \frac{1}{n} H_\mu(\xi_0^{n-1})$$

$$\leq \inf \frac{n}{n} \log p = \log p$$

"=" only if $\forall n$

$$\mu([x]_{\xi_0^{n-1}}) \equiv \frac{1}{p^n}$$

$\Rightarrow$ μ is Lebesgue

Markov shifts: G directed graph w/ V vertices, $n = \#V$

Assume G $\begin{cases} \text{s-ple} \\ \text{strongly connected} \end{cases}$

A adjacency matrix $= \begin{cases} A \text{ is } \{0,1\} \\ \forall\, i,k \ \exists n \ (A^n)_{ik} > 0 \end{cases}$

$A_{ij} = \#$ of directed edges $i \mapsto j$

$X =$ space bi-infinite paths on G $\circlearrowleft$ T shift

Ex: Full shift n-symbols $\hat{=}$ complete graph

Markov Measure:

- $P \leq A$ stochastic matrix ($\Leftrightarrow$ rows are prob vectors)

  P_{ij} prob of going from i to j

- p initial distribution

  p prob vector

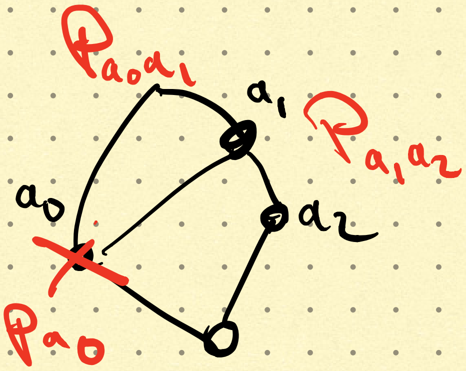
$$pP = p$$

$\mu = \mu_{P, p}$ def by

$$\mu\left([a_0, \dots, a_s]_m^{m+s} \right) = p_{a_0} \prod_{j=0}^{s-1} P_{a_j a_{j+1}}$$

cylinder on X

$$= \left\{ \underline{x} : \begin{array}{l} x_m = a_0 \\ \vdots \\ x_{m+s} = a_s \end{array} \right\}$$



$\Rightarrow$ extends to prob on X, shift-inv

<u>Parry measure</u>:

<u>Perron–Frobenius Thm</u>

$\Rightarrow$ A has a unique maximal positive eigenvalue λ

and $\exists$ left & right eigenvecs w/ strictly positive entries

$$\underline{u} A = A \underline{v} = \lambda A$$

Normalize $\underline{u}, \underline{v}$ st $\langle u, v \rangle = 1$

$\underline{p}$ st $p_i = u_i v_i$

$$P_{ij} = \frac{A_{ij} v_j}{\lambda v_i} \rightsquigarrow \underline{p} P = \underline{p}$$

$\xi = \{ [i]_0 \}$ 2-sided generator for shift T

C cylinder, $C \in \xi_0^m$ $C = [i_0, \dots, i_m]$ non-empty

μ = Parry measure $\mu_{P, p}$

Lemma:

$$\mu(C) = p_{i_0} \prod_{j<m} P_{i_j i_{j+1}}$$

$$= u_{i_0} v_{i_0} \prod_{0 \le j < m} \frac{\overbrace{A_{i_j i_{j+1}}}\, v_{i_{j+1}}}{\lambda v_{i_j}}$$

$$= u_{i_0} v_{i_0} \lambda^{-(m)} \frac{v_{i_m}}{v_{i_0}}$$

$$= \underbrace{u_{i_0} v_{i_m}}_{T} \lambda^{-(m)}$$

bnd by a cnst from above and below

<u>Lemma</u>: μ Markov measure

$$h_{\mu_{P,p}}(\sigma) = -\sum_{i,j} p_i P_{ij} \log P_{ij}$$

<u>Proof</u>: ξ generator $= \{ [i_0] \}$

$$\frac{1}{n} H_\mu(\xi_0^{n-1}) =$$

$$\frac{1}{n} - \sum_{\substack{C \in \xi_0^{n-1} \\ = [i_0, \dots, i_{n-1}]_0^{n-1}}} \left(p_{i_0} \prod_{j \le n-2} P_{i_j i_{j+1}} \right) \log \left(p_{i_0} \prod_{j \le n-2} P_{i_j i_{j+1}} \right)$$

$$= -\frac{1}{n} \sum_{i_0, i_1, \dots, i_{n-1}} p_{i_0} \log p_{i_0} \left(\prod_{j \le n-2} P_{i_j i_{j+1}} \right)$$

$+ \dots$

$$= -\frac{1}{n}\sum_{i_0} p_{i_0}\log p_{i_0}$$

$$-\frac{1}{n}\sum_{\substack{i_0, i_1, \dots, i_{n-1} \\ i_j, i_{j+1}}} p_{i_0}\sum_{j}\overline{\prod P_{i_k i_{k+1}}}\log P_{i_j i_{j+1}}$$

Fix j

$$\sum_{i_0} p_{i_0} P_{i_0 i_1} = p_{i_1}$$

$$= -\frac{1}{n}\sum_{i} p_i \log p_i \to 0$$

$$-\frac{1}{n}(n-1)\sum_{i_j, i_{j+1}} p_{i_j} P_{i_j i_{j+1}}\log P_{i_j i_{j+1}}$$

$$\frac{1}{n}(n-1) \to 1$$

<u>Cor</u>: μ Parry $\quad h_\mu(\sigma) = \log\lambda$ ☐

Thm: The Parry measure is the unique measure of maximal entropy.

Proof (Adler-Weiss):

Assume that ν has $h_\nu(T) \geq \log\lambda$

$\nu \perp \mu$ ie $\exists A_n \in \sigma(\xi_{-n}^{n})$

$$\mu(A_n) \to 0$$
$$\nu(A_n) \to 1$$

Denote $\# A_n = \#$ of ξ_{-n}^{n} atoms needed to make up A_n

$$(2n+1)\log\lambda \leq (2n+1)h_\nu(T) \leq H_\nu(\xi_0^{2n})$$
$$= H_\nu(\xi_{-n}^{n})$$
$$\leq H_\nu(\xi_{-n}^{n} \mid \{A_n, A_n^c\})$$
$$+ H_\nu(\{A_n, A_n^c\})$$

$$\leq \log 2 + \nu(A_n)\, H_{\nu_{A_n}}(\xi_{-n}^{n}) + \nu(A_n^c)\, H_{\nu_{A_n^c}}(\xi_{-n}^{n})$$

$$= \log 2 + \nu(A_n) \log \# A_n$$

$$+ \nu(A_n^c) \log \# A_n^c$$

1st Lemma $\mu(C) \asymp \lambda^{-(2n\ )}$, $C \in \xi_{-n}^{n}$

$$\Rightarrow \mu(A_n) \asymp \# A_n\, \lambda^{-(2n\ )}$$

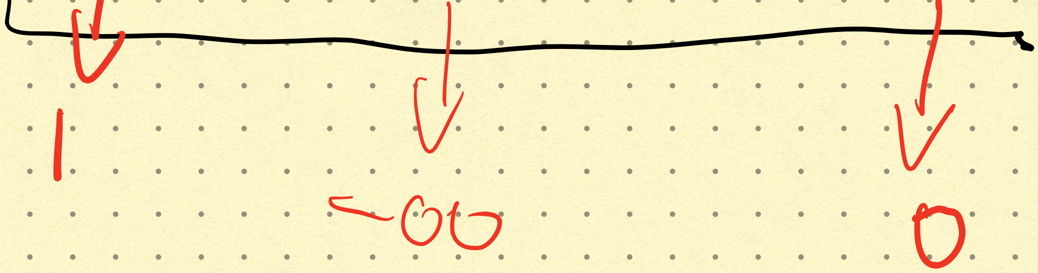
$$\geq K\, \# A_n\, \lambda^{-(2n\ )}$$

$$\leq \log 2 + \nu(A_n)\left(\log \frac{\mu(A_n)}{K} \lambda^{(2n\ )}\right)$$

$$+ \nu(A_n^c)\left(\log \frac{\mu(A_n^c)}{K} \lambda^{(2n\ )}\right)$$

$$= \log 2 - \log K + \underline{(2n\ )\log \lambda}$$

$$+ \nu(A_n) \underbrace{\log \mu(A_n)} + \nu(A_n^c) \underbrace{\log \mu(A_n^c)}$$



$\to -\infty$

$\blacksquare$