Lecture 10

$a \in G \qquad a \curvearrowright G/\Gamma \qquad \mu$ a-inv ergodic prob

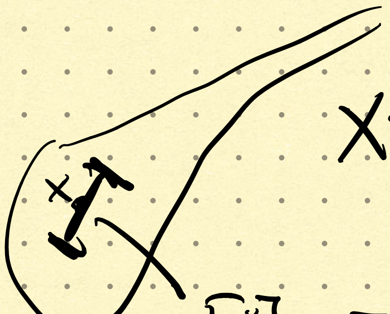
$G_a^- = \{ g : a^n g a^{-n} \to e \}$

Outline

$$h_\mu(a) \longleftrightarrow h_\mu(a, G_a^-)$$

$$\|$$

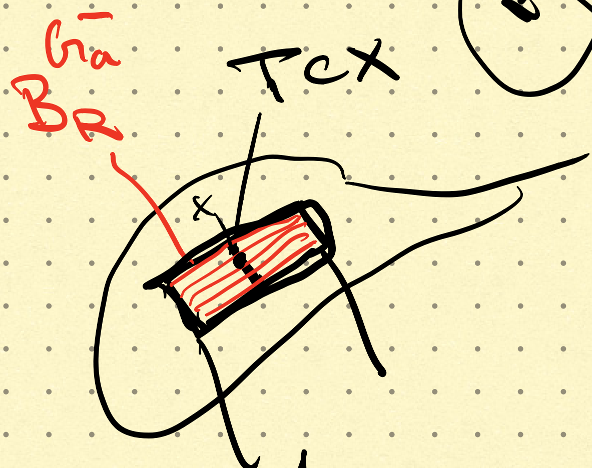
$$H_\mu(\mathcal{A} | a^{-1}\mathcal{A})$$



$X = G/\Gamma$

$[x]_{\mathcal{A}} = V_x \cdot x$

$V_x \subset G_a^-$



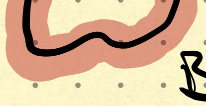
$\mathcal{A}_x$ supp $\underline{B_R^{G_a^-} \cdot T} \qquad [h.t]_{\mathcal{A}_x} = B_R^{G_a^-}.t$

$\mathrm{Lie}\, G_a^- \oplus V = \mathfrak{g} \qquad T = \exp(V \cap B_\varepsilon^{\mathfrak{g}}).x$

Lemma 1: μ ae x, leb ae $r > 0$ $\exists c = c_{x,r}$

$$\mu(\partial_\delta B_r(x)) \le c\delta \qquad \forall \delta > 0$$

where $\partial_\delta B = \{ x \in X : \inf_{z \in B} d(x,z) + \inf_{z \in B^c} d(x,z)$



B

$< \delta \xi$

Proof: $f(r) = \mu(B_r(x))$ monotone $\Rightarrow$ differentiable leb a.e. r

$$\frac{\mu(\partial_\delta B_r(x))}{2\delta} \le \frac{\mu\left(B_{r+\delta}^{(x)} \setminus B_{r-\delta}^{(x)}\right)}{2\delta}$$

$$= \frac{f(r+\delta) - f(r-\delta)}{2\delta}$$

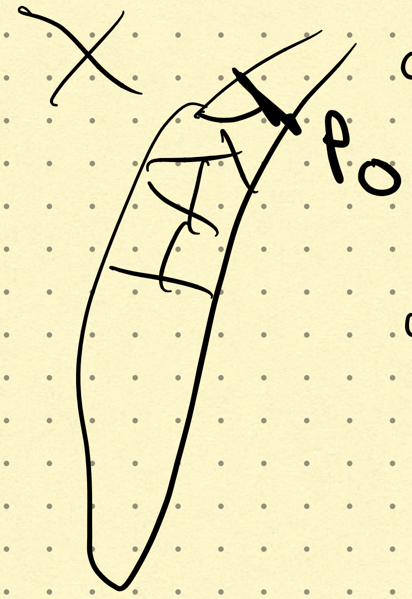
$$\to f'(r) \qquad \delta \to 0 \text{ a.e. } r$$

$$\le c \qquad \forall\, \delta \ll 1$$

$\forall\, \delta > 0$ μ prob ∎

Def: ξ finite partition say ξ has μ-thin boundary if $\mu\left(\bigcup_{P\in\xi} \partial_\delta P\right) \le c\delta$ for some c $\forall \delta$

Lemma 2: $\forall \varepsilon$ $\exists$ finite $\mathcal{P}$ w/ μ-thin boundary



and one $P_0 \in \mathcal{P}$ $\mu(P_0) < \varepsilon$

$X \setminus P_0$ compact

and all other $P \in \mathcal{P}$ $\operatorname{diam}(P) < r_{\text{inj rad}}$ on $X \setminus P_0$

Proof: $P_0 = B^c_{r_0}(x_0)$ $\quad x_0, r_0$ as in Lemma 1

and $\mu(P_0) < \varepsilon$

cover P_0^c by $B_r(x)$

$r <$ inj rad

(x, r) in Lemma 1

$X \setminus P_0$ compact $\rightsquigarrow \exists$ finite subcover

$\blacksquare$

Lemma 3: $\exists\, \alpha, d > 0$ $\forall r \leq 1$

$$a^n B_r^{G_{\bar{a}}} a^{-n} \subset B^{G_{\bar{a}}}_{d e^{-n\alpha} r}$$

⌜ Check on Lie alg level, Jordan normal form a ⌟

<u>Lemma 4</u>: If ξ finite partition μ-thin boundary then $\forall_{a.e.}\, x$ $\exists \delta$ so that $B_\delta^{G_a^-}.x \subset [x]_{\xi_0^\infty}$

<u>Proof</u>: $E_n^\delta = a^{-n}.\,\partial_{de^{-n\alpha}\delta}\,\xi$

$$\mu\Big(\bigcup_n E_n^\delta\Big) \ll \delta \quad (\xi\ \mu\text{-thin bndry})$$

$\Rightarrow$ a.e. x $\exists \delta > 0$ $x \notin E_n^\delta$ $\forall n$

Claim: For such x, the Lemma holds:

$u \in B_\delta^{G_a^-}$ If not: $u.x \notin [x]_{\xi_0^\infty}$

$\exists n$ $\quad a^n u.x \notin [a^n.x]_\xi$

$$a^n u.x = \underbrace{(a^n u a^{-n})}_{\in B_{\delta de^{-n\alpha}}} a^n x$$

$\Rightarrow$ $a^n.x \in \partial_{de^{-n\alpha}\delta}\xi \Rightarrow x \in E_n^\delta$ ↯ ∎

Def: $A \subset \mathcal{B}_X$ c.g. is U-subordinate

($U < G_a^-$ a-normalized) (mod μ)

if a.e. x $\exists \delta > 0$

$$B^U_\delta \cdot x \subset [x]_A \subset B^U_{\frac{1}{\delta}} \cdot x$$

a-decreasing: $a^{-1}A \subset A$.

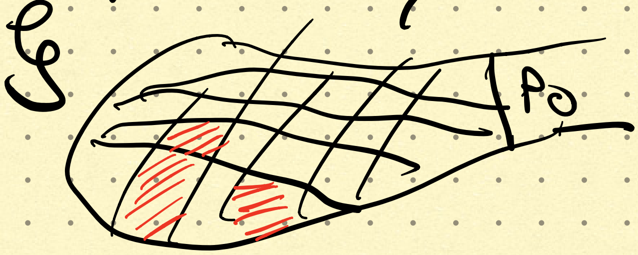
Prop: $\exists$ c.g. U-subordinate a-decreasing σ-algebra A

Proof: Take ξ from Lemma 2

$$\xi = \{ P_0, P_1, \dots, P_\kappa \}$$

P_0: small mass; $P_1, \dots, P_\kappa$: small diam

μ-th body



$P_\ell = B_\ell \Gamma$

$B_\ell \subset G$

$$\xi^U = \sigma\{ P_0, (B_\ell \cap A)\Gamma \}$$

$A \subset B^G_U$

$\ell \geq 1$

$A = (\xi^u)_0^\infty$ $\quad$ a-decreasing by construction

$\xi^u\big|_{X\setminus P_0}$ $\quad$ U - subordinate

Hence for $x \in X\setminus P_0$ $\quad [x]_A \subset [x]_{\xi^u} \subset B^U_{r_x}\cdot x$

$x \in P_0$: a ergodic

$\Rightarrow \exists n$ st $a^n . x \in X \setminus P_0$

$$[x]_A \subset [x]_{a^{-n}.A} = a^{-n}\,[a^n . x]_A$$

$$\underbrace{\in \left(a^{-n} B^U_{r_{a^n.x}} a^n\right)}_{\subset\; B^U_{R_{n,x}}} x$$

Remains to show $B^U_\delta . x \subset [x]_A$

Lemma 4 : $B^{G_a^-}_\delta . x \subset [x]_{\xi_0^\infty}$

If $u \in B_\delta^{\cancel{G_a^-}\, U}$, $y = u.x$

$\forall n \quad a^n . y \in [a^n . x]_{\xi} = P_{k(n)}$

• $k(n)=0$ $\quad [a^n.x]_{\xi^u} \ni a^n.y$

• $k(n) \geq 1$ $\quad a^n.y = \underbrace{(a^n u a^{-n})}_{\subset B^u_\delta} a^n.x$ $\qquad a^n x = g\Gamma$

$$= \underbrace{\left(B_{k(n)} \cap B^u_\delta g\right)}_{\in \mathcal{B}_{u(a)}|_{B_{k(n)}}} \Gamma$$

$$= [a^n.x]_{\xi^u} \qquad \blacksquare$$

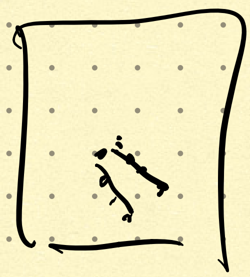
Prop: $\exists$ ξ generator st $\mathcal{A} = \xi_0^{\infty}$ $\quad$ ($\xi_{-\infty}^{\infty} = \mathcal{B}_X$)

G_a^--subordinate

Proof (hyperbolic toral auto A):

If $x, y \in \mathbb{T}^n$ close by

$$y = x + v^- + v^+$$



Apply A^n $n \in \mathbb{Z}$ then $A^n x, A^n y$ will be $\frac{1}{2}$-apart

2) For partitions ξ that are suff. fine

$\nexists$ separates pts

$\Rightarrow$ For such ξ , $\xi_{-\infty}^{\infty} = \mathcal{B}$ ∎

Prop: Suppose $\overline{B_R^U} \to X$ injective

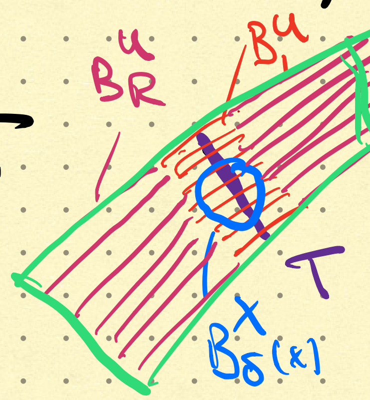
Fix x: $u \mapsto u.x$

then $\exists\, T \subset X$ "cross section" (or transversal)

st

- $\exists \delta \quad B_\delta^X(x) \subset B_1^U . T$
- $B_R^U \times T \to B_R^U . T$

  $(u,t) \mapsto u.t$ injective



and there σ-alg $\mathcal{A}_x$ on $B_R^U . T$

st $[u.t]_{\mathcal{A}_x} = B_R^U . t$

$\mathcal{A}_x$ is called (U,R)-flower with base $B_\delta^X(x)$

Proof: • Upgrade $u \mapsto u.x$ injective on $\overline{B^U_R}$

to $g \mapsto g.x$ injective on $\overline{B^U_R . B^G_\varepsilon}$

If not possible, $u^i_n g^i_n . x$ same pt

$g^i_n \in B^G_{\frac{1}{n}}$ $\downarrow$

U $u^i e.x$

u^i_n conv subseq

$\swarrow$

u^i

$\perp$

• $\operatorname{Lie} G = \operatorname{Lie} U \oplus V$

$T = \exp V \cap B^G_\varepsilon \cdot x$

$B^U_R \times T$ injective by 1st bullet pt

$B^U_1 . T \supset B^X_\delta(x)$

b/c $\operatorname{Lie} U \times V \to \exp w . \exp v$

(w, v) local diffeo $\mathfrak{g} \to G$

A_x image of $\{1, B^U_R\} \times B_T$

under $(u,t) \mapsto u.t$ $\blacksquare$

Prop: $\forall R>0$ $\exists \{x_n\}, \{\delta_n\}$ st $\exists$ (U, R) flowers A_{x_n} w/ basis $B^X_{\delta_n}(x_n)$

$$X = \bigcup B^X_{\delta_n}(x_n)$$

Proof: $U < G_a^-$ $\quad u \mapsto u.x$ injective on U

ae x

(if not $\quad u.x = x$

$$a^n u.x = a^n.x$$

$$(a^n u a^{-n})\, a^n.x$$

$e \Rightarrow$ Inj rad of $a^n.x \to 0$

$$a^n.x \to \infty$$

Poincare: divergent pts are null sets

ae x $\quad A_x$ (U, R) w/ base $B^X_{\delta_x}(x)$

cover X

X 2nd co-tbl ⇝ ∃ co-tble subcover

Definition: If $A \subset B_X$ $\mathcal{U}$-subordinate

$[x]_A = V_x . x$ $\quad$ $V_x \subset \mathcal{U}$ open bnd

$V_x . x$ is called a $\mathcal{U}$-plaque

Reminder: Leafwise measure $\mu_x^{\mathcal{U}}$

loc finite meas. on $\mathcal{U}$

$x = g\Gamma$ $\qquad$ $\mu_g = (\mu_G)_g^{\mathcal{B}_{\mathcal{U}} \backslash G}$ ← fibre measure

$\uparrow$

$\mathcal{U}g$

$\mu_x^{\mathcal{U}} = \mu_g . g^{-1}$ ← (defined up to ∼)

Push forward by mult by right of g^{-1}

Fix normalization $\mu_x^{\mathcal{U}}(B_1^{\mathcal{U}}) = 1$

Key proposition: $\mathcal{A}$ $\mathcal{U}$-subordinate

write $[x]_{\mathcal{A}} = V_x . x$

$V_x \subset \mathcal{U}$

$$\mu_x^{\mathcal{A}} = \frac{1}{\mu_x^{\mathcal{U}}(V_x)} \mu_x^{\mathcal{U}}\Big|_{V_x} . x$$

push forward wrt $a \mapsto a.x$

Proof: Cover X w/ $\cancel{\text{w/}}$ flowers $\mathcal{A}_{x_n}$

w/ $B_{\delta_n}(x_n)$

$\mathcal{A}_{x_n}$ σ-alg on a set $B_R^{\mathcal{U}} T$

$= B_R^{\mathcal{U}} T_G \Gamma$

atoms of $\mathcal{A}_{x_n}$

lift to G

$[u t_G \Gamma]_{\mathcal{A}_{x_n}} = B_R^{\mathcal{U}} t_G \Gamma$

We get the same if look $\pi_* \mathcal{B}_{\mathcal{U} \backslash G}\Big|_{B_R^{\mathcal{U}} T_G}$

$y \in B_1^{\mathcal{U}} T$ $\quad y = u t_G \Gamma$

proj $g \mapsto g\Gamma$

$$\Rightarrow \mu_y^{\mathcal{A}_{x_n}} = \pi_* \left(\mu_G\big|_{B_R^{\mathcal{U}} T_G}\right)_{u t_G}^{\mathcal{B}_{\mathcal{U} \backslash G}}$$

$$\sim \pi_*\left(\mu_{ut_G}\Big|_{B^u_R T_G}\right)$$ (used defining properties of fibre measure)

$$\sim \pi_*\left(\mu^u_{ut}\cdot ut_G\Big|_{B^u_R T_G}\right)$$ (def leafwise measure)

supp Uut_G

$$= \mu^u_y\Big|_{B^u_R u^{-1}}\cdot y$$

Put $\mathcal{C} = \mathcal{A}\big|_{B^u_R T} \vee \mathcal{A}_X$

since both σ-algebras have atoms that plaques

$\Rightarrow$ $\mathcal{C}$ atoms are also plaques

Both $A|_{B_R^u T}$-atoms and A_x-atoms

are countable union of $\mathcal{E}$-atoms

$$\Rightarrow \quad \mu_y^{A|_{B_R^u T}}\Big|_{[x]_{\mathcal{E}}} \sim \mu_y^{A_{x_n}}\Big|_{[x]_{\mathcal{E}}} \qquad \tilde{A}_2$$

$\uparrow$

$\tilde{A},\ \tilde{A} \subset \mathcal{E}$

then $\mu_z^{\tilde{A}}$ a.e. x $\quad \left(\mu_z^{\tilde{A}}\right)_x^{\mathcal{E}} = \mu_x^{\mathcal{E}}$

(general property)

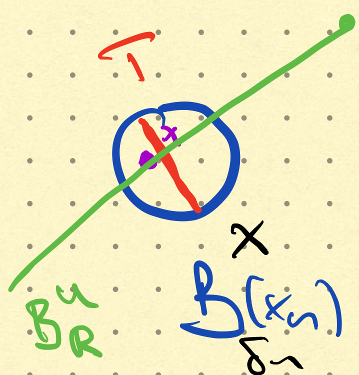
$\tilde{A}$ atoms are countably union of $\mathcal{E}$ atoms

$$\frac{1}{\mu_z^{\tilde{A}}([x]_{\mathcal{E}})}\, \mu_z^{\tilde{A}}\Big|_{[x]_{\mathcal{E}}}$$

$$\tilde{A} = A|_{B_R^u T} \text{ or } A_{x_n}$$

$$x \in B^X_{\delta_n}(x_n) \subset B_1^u T$$

$$B^u_{R-1} \circ x \subset [x]_{A_{x_n}}$$



$$V_x \cdot x = [x]_A$$

$$V_x \cap B^{U}_{R-1} \cdot x \subset [x]_e$$

Hence

$$\mu_x^A\Big|_{B^U_{R-1}\cdot x} \overset{\star_2}{\sim} \mu_x^{A\times U}\Big|_{V_x \cap B_{R-1}\cdot x}$$

$$\overset{(\star)}{\sim} \mu_x^U\Big|_{B_{R-1}\cap V_x} \cdot x$$

R arb $\nearrow \infty$

$\Rightarrow$ Prop

i.e. $\mu_x^A \sim \mu_x^U\Big|_{V_x} \cdot x$ $\blacksquare$

Thm C: $U < G_a^-$ a-normalized

μ ergodic $\quad \mathcal{A}$ U-subordinate a-decreasing σ-alg

- $H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A}) = h_\mu(a, U)$

$$\overset{\text{Def}}{=} \int \mathrm{vol}_\mu^U(a)(x)\, d\mu(x)$$

μ ae x $\quad = \mathrm{vol}_\mu^U(a)(x)$

$$\overset{\text{Def}}{=} \lim_{n\to\infty} \frac{1}{n} \log \mu_x^U(a^{-n} B_1^U a^n)$$

- $U' < U \quad h_\mu(a, U') \leq h_\mu(a, U)$

Thm A: $h_\mu(a, U) \leq h_\mu(a)$

and "$=$" if $U = G_a^-$

Thm C $\Rightarrow$ Thm A: Take $\mathcal{P}$ generator st $\mathcal{A} = \mathcal{P}_0^\infty$

G_a^--subordinate

$$\Rightarrow h_\mu(a) \overset{VS}{=} H_\mu(\xi \mid \xi_1^\infty)$$

$$= H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})$$

$$\overset{\text{Thm C}}{=} h_\mu(a, G_a^-)$$

$$\overset{\text{Thm C}}{\geq} h_\mu(a, G) \qquad \blacksquare$$

<u>Proof Thm C:</u>

- $$-\frac{1}{n} \log \mu_x^{a^{-n}\mathcal{A}}([x]_\mathcal{A})$$

$$\to H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})$$

$\ulcorner$ $a^{-1}\mathcal{A}$-atoms countable union of $\mathcal{A}$-atoms

$$\mu_x^{a^{-1}\mathcal{A}}\Big|_{[x]_\mathcal{A}} = \mu_x^{a^{-1}\mathcal{A}}([x]_\mathcal{A})\, \mu_x^{\mathcal{A}}$$

Rpt this we get

$$\mu_x^{\bar{a}^n A}([x]_A) = \prod_{k \le n} \mu_x^{\bar{a}^k A}([x]_{\bar{a}^{k+1} A})$$

Apply $-\frac{1}{n}\log$ get

$$\frac{1}{n} \sum I_\mu(\bar{a}^{k+1} A \mid \bar{a}^k A)(x)$$

$$= \frac{1}{n} \sum I_\mu(A \mid \bar{a}^1 A)(a^{k-1}.x)$$

$$\xrightarrow{\text{Ergodic Thm}} \int I_\mu(A \mid \bar{a}^1 A)$$

$$= H_\mu(A \mid \bar{a}^1 A)$$

$\perp$

- $\frac{1}{n} \log \mu_x^u(\bar{a}^n B_1^u a^n)$

(last time) $\to \mathrm{vol}_\mu^u(a \| (x)$

and can replace B_1^u

by B_r^u since $\exists k$

$a^{-k} B_1 a^k \subset B_r^u$

$\subset a^{-k\pm 1} B_1$

$a^{+k\mp 1}$

Combine:

$Y = \{ x \in X \quad B_\delta^u \cdot x \subset [x]_A \subset B_{\delta^{-1}}^u \cdot x \}$

(fix δ so that $\mu(Y) > 0$

μ ergodic: a.e. x

∞-many n st $a^n \cdot x \in Y$

For such x, n

$B_\delta^u \subset V_{a^n \cdot x} \subset B_{\delta^{-1}}^u$

$[x]_{a^{-n}A} = a^{-n} [a^n x]_A$

$= a^{-n} V_{a^n \cdot x} a^n \cdot x$

$$=: V_{n,x}$$

$$\rightsquigarrow \quad a^{-n} B^U_\delta a^n \subset V_{n,x} \subset a^{-n} B_{\delta^{-1}} a^n$$

<u>Key prop</u>: $\mu_x^{a^{-n}.A} \sim \mu_x^U\big|_{V_{n,x}} \cdot x$

Plug in $[x]_A$

$$\rightsquigarrow \quad \mu_x^{a^{-n}A}([x]_A) = \frac{\mu_x^U(V_x)}{\mu_x^U(V_{n,x})}$$

$$\mu_x^U(a^{-n} B_\delta a^n) \le \mu_x^U(V_{n,x}) \le \mu_x^U(a^{-n} B^U_{\delta^{-1}} a^n)$$

$$\|$$

$$\frac{\mu_x^U(V_x)}{\mu_x^{a^{-n}A}([x]_A)}$$

Take $\frac{1}{n}\log$

$$\boxed{h_\mu(a, \mathcal{U}) = H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})}\ h_\mu(a, \mathcal{U})$$

Thm C ∎